Subject: Statistics for Economics
Course Code:ECON4008
Topic: Discrete Probability Distributions
M.A. Economics (2nd Semester)

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Theoretical Distribution

In this topic, we will cover the following *univariate* probability distributions:

i. **Binomial Distribution**

ii. **Poisson Distribution**

iii. **Normal Distribution**

The first two distributions are *discrete* probability distributions and the third is a *continuous* distribution.

Note:

- **Discrete random variable:** Only takes finite or countable many number of values. For example, marks obtained by students in a test, the number of defective mangoes in a basket of mangoes, number of accidents taking place on a busy road, etc.
- **Continuous random variable:** The random variable assume infinite and uncountable set of values. In this case, we usually talk of the value in a particular interval and not at a point. For example, the age, height or weight of students in a class are all continuous random variable.

Binomial Distribution

Assumptions

i. n, the number of trials are finite.

ii. Each trial results in two mutually exclusive and exhaustive outcomes, termed as success and failure.

iii. Trials are independent.

iv. p, the probability of success is constant for each trial. Then $q=1-p$, is the probability of failure in any trial.

Binomial Distribution

Probability function of binomial distribution

- If X denotes the number of successes in n trials satisfying the assumptions, then X is a random variable which can take the values 0,1,2,…,n; since in n trials we may get no success (all failures), one success, two successes, …..or all the n successes.
- We are interested in finding the corresponding probabilities of 0,1,2,…., n successes. The general expression for the probability of r successes is given by:
- $$p(r) = P(X = r) = {}^{n}C_{r}\, p^{r} q^{n-r};\ r = 0,1,2,\ldots,n$$

$$Where\ {}^{n}C_{r} = \frac{n!}{(n-r)!\,r!}$$

Binomial Distribution

- Putting r=0,1,2,...,n, we get the probabilities of 0,1,2,...,n successes respectively in n trials and these are tabulated in the below table.

Binomial probabilities

r	p(r)=P(X=r)
0	${}^{n}C_{0}\,p^{0}.q^{n-0} = q^{n}$
1	${}^{n}C_{1}\,p^{1}.q^{n-1}$
2	${}^{n}C_{2}\,p^{2}.q^{n-2}$
.	.
.	.
.	.
n	${}^{n}C_{n}\,p^{n}.q^{n-n} = p^{n}$

Moments of Binomial Distribution

- For a Binomial random variable X ~ Binomial (n,p),
- Mean=E(X) = np;
- Var(X) $=\mu_2 = \sigma^2 = np(1-p) = npq$ since ($p+q=1$ & $1-p=q$)
- $\mu_3 = npq(q-p)$, $\mu_4 = npq[1+3pq(n-2)]$
- Moments coefficient of skewness is:

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{[npq(q-p)]^2}{(npq)^3} = \frac{(q-p)^2}{npq}$$

$$\gamma_1 = +\sqrt{\beta_1} = \frac{\mu_3}{\mu_2^{3/2}} = \frac{q-p}{\sqrt{npq}}$$

Moments of Binomial Distribution

- Moments coefficient of kurtosis is:

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{npq[1+3pq(n-2)]}{(npq)^2} = \frac{1+3pq(n-2)}{npq} = 3 + \frac{1-6pq}{npq}$$

$$\gamma_2 = \beta_2 - 3 = \frac{1-6pq}{npq}$$

Reference:

Gupta, S. C. (2015), *Fundamentals of Statistics*, Himalaya Publishing House.

Binomial Distribution

equal, each being np.

Example 14·1. *Ten unbiased coins are tossed simultaneously. Find the probability of obtaining,*

(*i*) *Exactly 6 heads* (*ii*) *At least 8 heads* (*iii*) *No head*

(*iv*) *At least one head* (*v*) *Not more than three heads* (*vi*) *At least 4 heads*

Solution. If p denotes the probability of a head, the $p = q = \frac{1}{2}$. Here $n = 10$. If the random variabl lenotes the number of heads, then by the Binomial probability law, the probability of r heads is given b

$$p(r) = P(X = r) = {}^nC_r\, p^r \cdot q^{n-r}$$

$$= {}^{10}C_r \left(\tfrac{1}{2}\right)^r \cdot \left(\tfrac{1}{2}\right)^{10-r} = {}^{10}C_r \cdot \left(\tfrac{1}{2}\right)^{10} = \frac{1}{1024}\,{}^{10}C_r \quad ..$$

(*i*) Required probability $= p(6) = \frac{1}{1024}\,{}^{10}C_6 = \frac{210}{1024} = \frac{105}{512}$ [From (*)]

(*ii*) Required probability $= P(X \geq 8) = p(8) + p(9) + p(10)$

$$= \frac{1}{1024}\left[{}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10}\right] \quad \text{[From (*)]}$$

$$= \frac{45 + 10 + 1}{1024} = \frac{56}{1024} = \frac{7}{128}$$

(*iii*) Required probability $= P(X = 0) = p(0) = \frac{1}{1024}\,{}^{10}C_0 = \frac{1}{1024}$ [From (*)]

(*iv*) Required probability $= P$[At least one head]

$$= 1 - P\,[\text{No head}] = 1 - p(0)$$

$$= 1 - \frac{1}{1024} = \frac{1023}{1024} \quad \text{[From Part (iii)]}$$

(*v*) Required probability $= P(X \leq 3) = p(0) + p(1) + p(2) + p(3)$

$$= \frac{1}{1024}\left[{}^{10}C_0 + {}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3\right] = \frac{1 + 10 + 45 + 120}{1024} = \frac{176}{1024} = \frac{11}{64}$$

(*vi*) Required probability $= (X \geq 4) = p(4) + p(5) + \ldots + p(10)$

$$= \frac{1}{1024}\left[{}^{10}C_4 + {}^{10}C_5 + \ldots + {}^{10}C_{10}\right]$$

Last part can be conveniently done as follows :

Required probability $= P(X \geq 4) = 1 - P\,(X \leq 3)$

$$= 1 - \left[p(0) + p(1) + p(2) + p(3)\right] = 1 - \frac{11}{64} = \frac{53}{64} \quad \text{[From P}$$

Binomial Distribution

Example 14·4. *Suppose that a Central University has to form a committee of 5 members from a list of 20 candidates out of whom 12 are teachers and 8 are students. If the members of the committee are selected at random, what is the probability that the majority of the committee members are students ?*

[*Delhi Univ. B.A. (Econ. Hons.), 2009*]

Solution. In the usual notations we have : $n = 5$;

$$p = \text{Probability of selecting a student member} = \frac{8}{20} = \frac{2}{5}$$

$$\Rightarrow \qquad q = \text{Probability of selecting a teacher member} = \frac{12}{20} = \frac{3}{5}$$

Let X denote the number of students selected in the committee. Then $X \sim B(n = 5, p = 2/5)$. Hence, by binomial probability distribution,

$$P(X = r) = p(r) = \binom{5}{r} \left(\frac{2}{5}\right)^r \left(\frac{3}{5}\right)^{5-r} \quad ; \quad r = 0, 1, 2, 3, 4, 5 \qquad \ldots(1)$$

The required probability is given by :

$$P(X \geq 3) = p(3) + p(4) + p(5) = \binom{5}{3} \left(\frac{2}{5}\right)^3 \cdot \left(\frac{3}{5}\right)^2 + \binom{5}{4} \left(\frac{2}{5}\right)^4 \cdot \left(\frac{3}{5}\right) + \binom{5}{5} \left(\frac{2}{5}\right)^5$$

$$= \frac{1}{5^5} [10 \times 8 \times 9 + 5 \times 16 \times 3 + 1 \times 32] = \frac{720 + 240 + 32}{3125} = \frac{992}{3125} = 0.3174$$

Binomial Distribution

Example 14·13. *Four fair coins are tossed. X is the number of heads that occur.*
(i) Find the probability function of X
(ii) Find the probability distribution of X and
(iii) Calculate the mean and standard deviation of X. [*Delhi Univ. B.A. (Econ.Hons.), 1999*]

Solution. (*i*) In the usual notations, $X \sim B\,(n, p)$ with $n = 4$ and

$$p = \text{Probability of head in toss of fair coin} = \frac{1}{2} \quad \Rightarrow \quad q = 1 - p = \frac{1}{2}$$

By binomial probability law, the probability function of X, (the number of heads obtained) is given by :

$$p(x) = P(X = x) = {}^nC_x\, p^x q^{n-x}$$

$$= {}^4C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{4-x} = {}^4C_x \cdot \left(\frac{1}{2}\right)^4 = \frac{1}{16} \cdot {}^4C_x \quad ; \quad x = 0, 1, \ldots, 4 \qquad \ldots(*)$$

(*ii*) Using (*), the probability distribution of X is given by :

x	0	1	2	3	4
$p(x)$	$\frac{1}{16} \times {}^4C_0$ $= \frac{1}{16}$	$\frac{1}{16} \times {}^4C_1$ $= \frac{4}{16} = \frac{1}{4}$	$\frac{1}{16} \times {}^4C_2$ $= \frac{1}{16} \times \frac{4 \times 3}{2} = \frac{3}{8}$	$\frac{1}{16} \times {}^4C_3$ $= \frac{4}{16} = \frac{1}{4}$	$\frac{1}{16} \times {}^4C_4$ $= \frac{1}{16}$

(*iii*) *Mean and s.d. of X.* Since $X \sim B\left(n = 4, p = \frac{1}{2}\right)$,

$$\text{Mean} = np = 4 \times \frac{1}{2} = 2 \quad ; \quad \text{Variance } (\sigma^2) = npq = 4 \times \frac{1}{2} \times \frac{1}{2} = 1 \quad \Rightarrow \quad \text{s.d. } (\sigma) = 1.$$

Poisson Distribution

- Poisson distribution was derived in 1837by a French mathematician Simen D. Poisson (1781-1840). Poisson distribution may be obtained as a limiting case of Binomial probability distribution under the following conditions:

  (i) n, the number of trials is indefinitely large i.e. $n \to \infty$,

  (ii) p, the constant probability of success for each trial is indefinitely small i.e. $p \to 0$.

  (iii) np=m

  Under the above three conditions the binomial probability function tends to the probability function of the Poisson distribution given below:

$$p(r) = P(X = r) = \frac{e^{-m}.m^{r}}{r!}, r = 1,2,3,....$$

  Where X is the number of successes, m=np and e=2.71828 [The base of the system of Natural Logarithms]

Poisson Distribution

Putting r=0,1,2,3,…., we obtain the probabilities of 0,1,2,3,… successes respectively, which are given in the below table.

r	p(r)=P(X=r)
0	$\frac{e^{-m}.m^0}{0!} = e^m$
1	$\frac{e^{-m}.m^1}{1!}$
2	$\frac{e^{-m}.m^2}{2!}$
3	$\frac{e^{-m}.m^3}{3!}$
.	.
.	.
.	.

The total probability is 1

$$\sum_{r-0}^{\infty} p(r) = e^{-m}[1 + m + \frac{m^2}{2!} + \frac{m^3}{3!} + \ldots.] = e^{-m} + e^m = e^{-m+m} = e^0 = 1$$

Poisson Distribution

Utility or importance of Poisson distribution

- Poisson distribution can be used to explain the behavior of the discrete random variables where the probability of occurrence of the event is very small and the total number of possible cases is sufficiently large.
- Some of practical situations where Poisson distribution can be used.

1. The number of telephone calls arriving at a telephone switch board in unit time (say, per minute).
2. The number of defects per unit of manufactured product.
3. To count number of bacteria per unit (Biology).
4. The number of accidents taking place per day on a busy road.
5. The number of typological errors per page in a typed material.

Constants and moments of Poisson Distribution

- Mean=m
- Variance=μ_2=m
- In the Poisson distribution, the mean=variance=m
- μ_3=m, μ_4=m+3m^2

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{m^2}{m^3} = \frac{1}{m} \qquad \gamma_1 = +\sqrt{\beta_1} = \frac{1}{\sqrt{m}}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{m+3m^2}{m^2} = 3 + \frac{1}{m} \qquad \gamma_2 = \beta_2 - 3 = \frac{1}{m}$$

$$if\, , m \to \infty, \beta_1 \to 0, \gamma_1 \to 0, \beta_2 \to 3, and \gamma_2 \to 0$$

Poisson Distribution

Example 14·20. *Between the hours 2 P.M. and 4 P.M. the average number of phone calls per minute coming into the switch board of a company is 2·35. Find the probability that during one particular minute there will be at most 2 phone calls. [Given $e^{-2\cdot35} = 0\cdot095374$]*

Solution. If the random variable X denotes the number of telephone calls per minute, then X wi follow Poisson distribution with parameter $m = 2\cdot35$ and probability function :

$$P(X = r) = \frac{e^{-m} \cdot m^r}{r\,!} = \frac{e^{-2\cdot35} \times (2\cdot35)^r}{r\,!} \quad ; \quad r = 0, 1, 2 \ldots \qquad \ldots($$

The probability that during one particular minute there will be at most 2 phone calls is given by :

$$P(X \le 2) = P(X = 0) + P(X = 1) + P(X = 2) = e^{-2\cdot35}\left(1 + 2\cdot35 + \frac{(2\cdot35)^2}{2\,!}\right) \qquad \text{[From (}$$

$$= 0\cdot095374 \times (1 + 2\cdot35 + 2\cdot76125) = 0\cdot095374 \times 6\cdot11125 = 0\cdot5828543.$$

Poisson Distribution

$= 0\cdot0183 \times 23\cdot67 = 0\cdot4332.$

Example 14·22. *If 5% of the electric bulbs manufactured by a company are defective, use Poisson distribution to find the probability that in a sample of 100 bulbs*

(*i*) *none is defective,*

(*ii*) *5 bulbs will be defective.* (*Given : $e^{-5} = 0\cdot007$*).

Solution. Here we are given :

$$n = 100,$$

$$p = \text{Probability of a defective bulb} = 5\% = 0\cdot05$$

Since p is small and n is large, we may approximate the given distribution by Poisson distribution. Hence, the parameter m of the Poisson distribution is :

$$m = np = 100 \times 0\cdot05 = 5$$

Let the random variable X denote the number of defective bulbs in a sample of 100. Then (by Poisson probability law),

$$P(X = r) = \frac{e^{-m}\, m^r}{r\,!} = \frac{e^{-5}\, 5^r}{r\,!};\ r = 0, 1, 2, \ldots \qquad (*)$$

(*i*) The probability that none of the bulbs is defective is given by :

$$P(X = 0) = e^{-5} = 0\cdot007 \qquad \text{[From (*)]}$$

(*ii*) The probability of 5 defective bulbs is given by :

$$P(X = 5) = \frac{e^{-5} \times 5^5}{5\,!} = \frac{0\cdot007 \times 625}{24} = \frac{4\cdot375}{24} = 0\cdot1823.$$

Poisson Distribution

Example 14·32. *Fit a Poisson distribution to the following data and calculate the theoretical frequencies.*

x	:	0	1	2	3	4
f	:	123	59	14	3	1

Solution.

x	0	1	2	3	4	
f	123	59	14	3	1	$\Sigma f = 200$
fx	0	59	28	9	4	$\Sigma fx = 100$

$$\therefore \quad \bar{x} = \frac{\Sigma fx}{\Sigma f} = \frac{100}{200} = 0{\cdot}5.$$

Thus, the mean (m) of the theoretical (Poisson) distribution is $m = \bar{x} = 0{\cdot}5$. By Poisson probabil[ity] law, the theoretical frequencies are given by :

$$f(r) = Np(r) = 200.\frac{e^{-m}\, m^r}{r\,!} \quad ; \quad r = 0, 1, 2, 3, \ldots$$

$$\therefore \quad f(0) = N\,p(0) = 200 \times e^{-m} = 200 \times e^{-0{\cdot}5} = 200 \times 0{\cdot}6065 = 121{\cdot}3.$$

TABLE 14·7 : **COMPUTATION OF EXPECTED FREQUENCIES**

x	*Expected Poisson Frequencies N.p(x)*			
0	$N\,p(0) = 121{\cdot}3$			$\simeq 121$
1	$N\,p(1) = N\,p(0) \times m$	$= 121{\cdot}3 \times 0{\cdot}5$	$= 60{\cdot}65$	$\simeq 61$
2	$N\,p(2) = N\,p(1) \times \frac{m}{2}$	$= \frac{60{\cdot}65 \times 0{\cdot}5}{2}$	$= 15{\cdot}3125$	$\simeq 15$
3	$N\,p(3) = N\,p(2) \times \frac{m}{3}$	$= \frac{15{\cdot}3125 \times 0{\cdot}5}{3}$	$= 2{\cdot}552$	$\simeq 3$
4	$N\,p(4) = N\,p(3) \times \frac{m}{4}$	$= \frac{2{\cdot}552 \times 0{\cdot}5}{4}$	$= 0{\cdot}32$	$\simeq 0$
Total				200

Thank you..