Definition :- (Integral Equation) (1)

An integral equation is an equation in which an unknown function appears under an integral sign. Integral equations play an important role in the study of initial and boundary value problems.

e.g. radiations transfer problem, neutron diffusion problem lead us to the integral equations, mechanics, applied mathematics and mathematical physics

(1) For $a \leq x \leq b$, $a \leq t \leq b$, then

$$\int_a^b k(x,t)\, y(t)\, dt = f(x)$$

$$y(x) - \lambda \int_a^b k(x,t)\, y(t)\, dt = f(x)$$

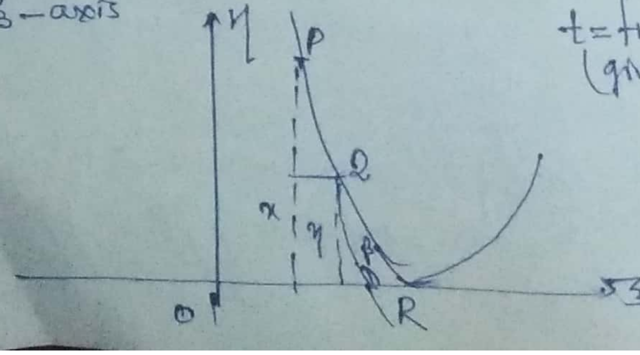
$$y(x) = \int_a^b k(x,t)\, [y(t)]^2\, dt$$

Abel's problem :-

Let us consider a smooth curve in a vertical plane (ξ, η). Suppose a particle starts from rest at any point P with ordinate x under the influence of gravity along the curve. Let t be the time taken by the particle from P to the lowest point R on the ξ-axis.

Let $t = f_1(x)$, where $f_1(x)$ is a given function. The absolute velocity of the particle at point Q on the curve whose ordinate is η, is $v = \sqrt{2g(x-\eta)}$ (given)

Let β be the angle of the inclination of the tangent to the curve at Q with the ξ-axis



$t = f_1(x)$ (given)

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then $\frac{d\eta}{dt} = -\sqrt{2g(x-\eta)}\,\sin\beta$

$$dt = -\frac{d\eta}{\sin\beta\sqrt{2g(x-\eta)}}$$

let $\frac{1}{\sin\beta} = \phi(\eta)$, then

$$\sqrt{2g}\,dt = -\frac{\phi(\eta)\,d\eta}{\sqrt{(x-\eta)}}$$

$$\sqrt{2g}\int_P^R dt = -\int_P^R \frac{\phi(\eta)\,d\eta}{\sqrt{(x-\eta)}}$$

$$\sqrt{2g}\,t = -\int_x^0 \frac{\phi(\eta)\,d\eta}{\sqrt{x-\eta}}$$

$$\sqrt{2g}\,f_1(x) = \int_0^x \frac{\phi(\eta)\,d\eta}{\sqrt{x-\eta}}$$

$$f(x) = \int_0^x \frac{\phi(\eta)\,d\eta}{\sqrt{x-\eta}}$$

where $f(x) = f_1(x)\sqrt{2g}$ is known and $\phi(\eta)$ is the unknown function

Find the equation of the curve as follows $\phi(\eta) = \frac{1}{\sin\beta}$

$$\Rightarrow \eta = \phi(\beta)$$

$$\because \frac{d\eta}{d\xi} = \tan\beta$$

Hence, the Abel's problem reduces to a solution of the Volterra integral equation of the ~~kind~~ first kind

$$f(x) = \int_0^x K(x,t)\,\phi(t)\,dt$$

where $\phi(x)$ is an unknown function, $K(x,t) = \frac{1}{\sqrt{x-t}}$ and $f(x)$ are given functions

The most general linear integral equation is of the form

$$g(x)y(x) = f(x) + \lambda\int_a^b K(x,t)\,y(t)\,dt \quad \text{—①}$$

where $f(x)$, $g(x)$ and $K(x,t)$ are known functions while $y(x)$ is to be determined λ is a non-zero real or real or complex parameter. The function $k(x,t)$ is called the kernel of the integral equation.

If a is a constant and $b=x$, the equation (1) is called Volterra Integral equation.

$$g(x)y(x) = f(x) + \lambda \int_a^x k(x,t)y(t)\,dt \quad (2)$$

if the limits of integration a, b are both constant then the equation (1) is called the Fredholm integral equation

Volterra Integral equation:-

A linear integral equation of the form

$$g(x)y(x) = f(x) + \lambda \int_a^x k(x,t)y(t)\,dt \quad (1)$$

is called a Volterra integral equation of the third kind

if $g(x)=0$, then eq (1) takes the form

$$f(x) + \lambda \int_a^x k(x,t)\, y(t)\,dt = 0 \quad (2)$$

and is called the Volterra integral equation of the first kind.

On the other hand, if $g(x)=1$, then the integral equation

$$y(x) = f(x) + \lambda \int_a^x k(x,t)y(t)\,dt \quad (3)$$

is known as the Volterra integral equation of the second kind.

In particular, if $f(x)=0$ in equation (3), then the integral equation

$$y(x) = \lambda \int_a^x k(x,t)\, y(t)\,dt$$

is called the homogeneous Volterra integral equation of the second kind.

Singular integral equation:-

A linear integral equation is called singular integral equation when one or both the limits of the integration are infinite or when the kernel becomes infinite over the range of integration

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For example! $y(x) = f(x) + \lambda \int_{-\infty}^{\infty} e^{-|x-t|} y(t)\,dt$ (4)

and $f(x) = \int_0^x \frac{y(t)}{(x-t)^\alpha}\,dt, \quad 0<\alpha<1.$

are Singular integral equations

Leibnitz rule of differentiation under integral Sign!.

Let $F(x,t)$ and $\frac{\partial F}{\partial x}$ be continuous functions of both x and t and let first order derivatives of $G(x)$ and $H(x)$ be continuous.

Then

$$\frac{d}{dx}\int_{G(x)}^{H(x)} F(x,t)\,dt = \int_{G(x)}^{H(x)} \frac{\partial F(x,t)}{\partial x}\,dt + F(x,H(x))\frac{dH}{dx}$$
$$- F(x,G(x))\frac{dG}{dx}$$

In particular, if $G(x)$ and $H(x)$ are constants then

$$\frac{d}{dx}\int_{G(x)}^{H(x)} F(x,t)\,dt = \int_{G(x)}^{H(x)} \frac{\partial F(x,t)}{\partial x}\,dt.$$

Important result for converting a multiple integral into a Single Integral!-

$$\int_{x_0}^{x} dx \int_{x_0}^{x} dx \cdots \int_{x_0}^{x} y(x)\,dx = \frac{1}{(n-1)!}\int_{x_0}^{x} (x-t)^{n-1} y(t)\,dt$$

where n is a positive integer and x_0 is a constant

Proof!. Let $I_n(x) = \int_{x_0}^{x} (x-t)^{n-1} y(t)\,dt$ — (1)

putting $n=1$ in above eqn (1), we get

$I_1(x) = \int_{x_0}^{x} 1.y(t)\,dt$ (By Leibnitz rule)

$$\frac{dI_1}{dx} = \int_{x_0}^{x} \frac{\partial}{\partial x} y(t)\,dt + y(x)\frac{dx}{dx} - y(x_0)\frac{dx_0}{dx}$$
$$= 0 + y(x) + 0$$
$$\frac{dI_1}{dx} = y(x) \quad — (2)$$

Next, $$\frac{dI_n}{dx} = \int_{x_0}^{x}\left[\frac{\partial}{\partial x}(x-t)^{n-1}\right] y(t)\,dt + (x-x)^{n-1} y(x)\frac{dx}{dx} - (x-x_0)^{n-1} y(x_0)\frac{dx_0}{dx}$$

$$\frac{dI_n}{dx} = (n-1)\int_{x_0}^{x} (x-t)^{n-2} y(t)\,dt + 0 - 0$$

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