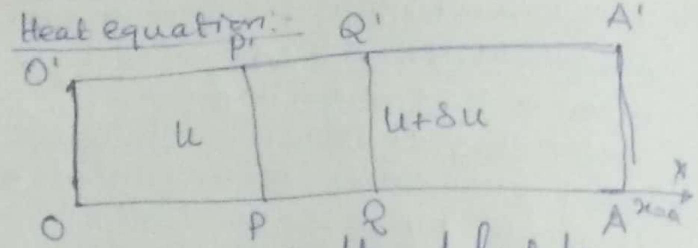
Derivation of one-dimensional Heat equation: ①



Let us consider the flow of heat by conduction in a rod of length a and of uniform section, the diameter of which is small in comparison with the radius of curvature. Suppose that the sides of the rods are insulated so that there is no radiation from the sides.

Consider an element of rod i.e. $PQP'Q'$.

Suppose that the rod is ~~heated~~ raised to an assigned temperature distribution at time $t=0$ and then heat is allowed to flow by conduction. To find the temperature $u(x,t)$ at any point x and at any time $t>0$.

Now, according to the Fourier law of heat conduction "The amount of the heat crossing through an area is proportional to the area and to the temperature gradient normal to the area"

i.e. The heat flux through area A normal to x-axis is $= -kA\frac{\partial u}{\partial x}$ —①

where u is the temperature

$\frac{\partial u}{\partial x}$ is temperature gradient normal to A and K is proportionality constant called thermal conductivity.

The minus sign represent heat flows in the direction of decreasing temperature. i.e. The heat is transferred in the direction opposite to the temperature gradient.

Using eqn ① the quantity of heat flowing into the element across the section PP' in time δt

$$= -kA\left(\frac{\partial u}{\partial x}\right)_x \delta t$$

Similarly, The quantity of heat flowing out of the element across the section QQ' in time δt $= -kA\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}\delta t$

Hence, The quantity of heat retained by the element is

$$= -kA\left(\frac{\partial u}{\partial x}\right)_x \delta t - \left[-kA\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}\delta t\right]$$

$$= KA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right]\delta t \quad (2)$$

Suppose that the above heat raises the temperature of the element by a small quantity δu, then the same amount of heat gained (or lost) by the element of mass $m = \rho A \delta x$, ρ being the density, the undergoes temperature change δu is given by

$$(\rho A \delta x)\sigma \delta u \quad (2)$$

where σ is the specific heat of the rod.

$\because$ Eqn (1) & (2) are equal, then we obtain

$$KA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right]\delta t = (\rho A \delta x)\sigma \delta u$$

Using Taylor's series ~~extension~~ expansion and neglecting term of order $(\delta x)^2$ and higher, we get

$$K\left[\frac{\frac{\partial u(x+\delta x,t)}{\partial x} - \frac{\partial u(x,t)}{\partial x}}{\delta x}\right] = \rho\sigma \frac{\delta u}{\delta t}$$

Now as $\delta x \to 0$, $\delta t \to 0$, we get

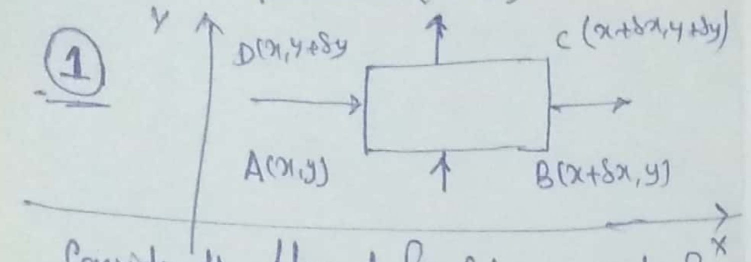
$$K\frac{\partial^2 u}{\partial x^2} = \rho\sigma\frac{\partial u}{\partial t}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{h}\frac{\partial u}{\partial t}$$

where $h = \frac{K}{\rho\sigma}$ is called thermal diffusivity and is assumed to be constant.

Laplace Equation (2D)

(1)



Consider the flow of heat in a metal plate of uniform thickness α (cm), density ρ (g/cm³), specific heat s (cal/g deg) and thermal conductivity k (cal/cm sec deg). Let the XY-plane be taken in one face of the plate. If the temperature at any point is independent of the z-coordinate and depends only on x and y and time t, then the flow is said to be two-dimensional. In this case, the heat flow is in the XY-plane only and is zero along the normal to the XY-plane.

Consider a rectangular element ABCD of the plate with sides δx and δy as shown in figure. By Fourier's law, the amount of heat entering the element in 1 sec from the side AB is $= -k\alpha\,\delta x\left(\frac{\partial u}{\partial y}\right)_y$

$$= -k\alpha\,\delta x\left(\frac{\partial u}{\partial y}\right)_y \quad \text{—(1)}$$

and the amount of heat entering the element in 1 sec from the side AD is

$$= -k\alpha\,\delta y\left(\frac{\partial u}{\partial x}\right)_x \quad \text{—(2)}$$

The quantity of heat flowing out through the side CD in 1 sec is

$$= -k\alpha\,\delta x\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} \quad \text{—(3)}$$

and the quantity of heat flowing out through the side BC in 1 sec is

$$= -k\alpha\,\delta y\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} \quad \text{—(4)}$$

Therefore, the total gain of heat by the rectangular element ABCD in 1 sec is

$$= -k\alpha\,\delta x\left(\frac{\partial u}{\partial y}\right)_y - k\alpha\,\delta y\left(\frac{\partial u}{\partial x}\right)_x + k\alpha\,\delta x\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} + k\alpha\,\delta y\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$$

$$= k\alpha\,\delta x\,\delta y\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x} + \frac{\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} - \left(\frac{\partial u}{\partial y}\right)_y}{\delta y}\right] \quad \text{—(5)}$$

(2)

The rate of gain of heat by the element is $= \rho\,\delta x\,\delta y\,\alpha s\,\frac{\partial u}{\partial t}$ —(6)

~~Thus, equating eqn (5) & (6)~~

Now from eqn (5) & (6), we obtain

$$\rho\,\delta x\,\delta y\,\alpha s\frac{\partial u}{\partial t} = k\alpha\,\delta x\,\delta y\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x} + \frac{\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} - \left(\frac{\partial u}{\partial y}\right)_y}{\delta y}\right]$$

Dividing both sides by $\alpha\,\delta x\,\delta y$ and taking limits as $\delta x \to 0$, $\delta y \to 0$, we get

$$\rho s\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial u}{\partial t} \quad \text{—(7)}$$

where $c^2 = k/\rho s$ is the diffusivity coefficient.

Eqn (7) gives the temperature of the plate in the transient state. In the steady state, u is independent of t, so that $u_t = 0$ $\left(\frac{\partial u}{\partial t} = 0\right)$ and the equation (7) reduces to

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{—(8)}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

which is the well-known Laplace equation in two dimensions.