# Quotient Topology

DR. SHEO KUMAR SINGH

Assistant Professor,

Department of Mathematics

School of Physical Sciences,

Mahatma Gandhi Central University, Motihari, Bihar-845401.

Email: sheokumarsingh@mgcub.ac.in

# Quotient Topology

Let $X$ be a topological space and $Y$ be a set. Let $q: \mathrm{X} \rightarrow Y$ be an onto map. Then

- the collection $\tau = \{V \subseteq Y \mid q^{-1}(V) \text{ is open in } X\}$ turns out to be a topology on $Y$ (verify it !!).

- this topology $\tau$ is said to be the quotient topology on $Y$ and such a map $q$ is said to be the quotient map.

- this topology is also called as the quotient topology on $Y$ induced by the quotient map $q$.

# Some examples of quotient topology

Example 1: Consider $\mathbb{R}$ with usual topology and a map

$q: \mathbb{R} \to \{a, b, c\}$, defined by $q(x) = \begin{cases} a, \text{if } x < 0 \\ b, \text{if } x = 0 \\ c, \text{if } x > 0 \end{cases}$.

Then $\{\emptyset, \{a, b, c\}, \{a\}, \{c\}, \{a, c\}\}$ is the quotient topology on $\{a, b, c\}$ induced by the map q.

Example 2: Consider $\mathbb{R}$ with usual topology and a map $q: \mathbb{R} \to \mathbb{Z}$, defined by

$$q(x) = \begin{cases} x, \text{if } x \in \mathbb{Z} \\ n, \text{if } x \in (n-1, n+1) \text{for an odd } n \in \mathbb{Z}. \end{cases}$$

Then the quotient topology on $\mathbb{Z}$ induced by $q$ is generated by the base $\mathfrak{B} = \{B(n) | n \in \mathbb{Z}\}$, where

$$B(n) = \begin{cases} \{n\}, \text{if } n \text{ is odd} \\ \{n-1, n, n+1\}, \text{if } n \text{ is even} \end{cases}$$

# Quotient space: construction of new spaces from a given topological space

Let

- $X$ be a topological space and $\sim$ be an equivalence relation on X.
- $\bar{X} = \{\bar{x} \mid \bar{x} \text{ is an equivalence class containing } x \in X\}$ (or equivalently, $\bar{X}$ is a partition of $X$).
- $q: X \to \bar{X}$ be the quotient map, defined by $q(x) = \bar{x}$.

Then

- $\bar{X}$ with resp. to the quotient topology is said to be a *quotient space* of $X$.

# Some examples of quotient spaces

Example 3: (Cylinder) Let $X = [0,1] \times [0,1]$ be the subspace of $\mathbb{R}^2$ and $\bar{X}$ be a partition of $X$ consisting of the sets $\{ \{(x, y)\} \mid 0 < x < 1 \text{ and } 0 < y < 1\}$ and sets $\{ \{(0, y), (1, y)\} \mid 0 \leq y \leq 1\}$. Then the quotient space $\bar{X}$ of $X$ is a cylinder of height 1.

Example 4: (Torus) Let $X$ as above and $\bar{X}$ be a partition of $X$ consisting of the sets $\{ \{(x, y)\} \mid 0 < x < 1 \text{ and } 0 < y < 1\}$, $\{ \{(x, 0), (x, 1)\} \mid 0 < x < 1\}$, $\{ \{(0, y), (1, y)\} \mid 0 < y < 1\}$ and the set $\{(0,0), (0,1), (1,0), (1,1)\}$. Then the quotient space $\bar{X}$ of $X$ is a torus.

# References:


- James R. Munkres, *Topology*, 2nd ed., PHI.
- Colin Adams and Robert Franzosa, *Introduction to Topology: Pure and Applied*, Pearson.

*THANK YOU*