Triangular Form

Let V be an n-dimensional vector space and T be a linear operator on V. Suppose T is represented by the triangular matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ 0 & 0 & \ddots & \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

Then characteristic polynomial of T is given by

$$Ch_A(x) = \det(xI-A) = (x-a_{11})(x-a_{22})\cdots(x-a_{nn}).$$

It can be easily shown ~~that~~ by induction that minimal polynomial are same as its characteristic polynomial.

Result: Suppose W is an invariant subspace under a linear operator $T: V \to V$. Then T induces a linear operator $\bar{T}$ on $\frac{V}{W}$ defined by

$$\bar{T}(v+W) = T(v)+W, \quad v \in V.$$

Moreover, T and $\bar{T}$ are zero of same polynomial. Thus the minimal polynomial of $\bar{T}$ divides the minimal polynomial of T.

Soln: Given W is invariant under T.

Suppose $u+W = v+W \Rightarrow u-v \in W$

$\Rightarrow T(u-v) \in W$

$\Rightarrow T(u)-T(v) \in W$

$\Rightarrow T(u)+W = T(v)+W$

Now $\bar{T}(u+W) = T(u)+W = T(v)+W = \bar{T}(v+W)$

So $\bar{T}$ is well-defined.

Clearly, we show that $\bar{T}$ is linear.

Since T is a linear operator on V, then T^n is also linear operator on V.

Again, W is T-invariant, implies that W is T^n-invariant.

Now, let $v + W \in \frac{V}{W}$ be arbitrary.

$$\overline{T^2}(v+W) = T^2(v) + W$$
$$= T(T(v)) + W$$
$$= \bar{T}(T(v) + W)$$
$$= \bar{T}(\bar{T}(v+W))$$
$$\Rightarrow \overline{T^2}(v+W) = \bar{T}^2(v+W).$$
$$\therefore \quad \bar{T}^2 = \overline{T^2}$$

Similarly $\bar{T}^n = \overline{T^n}$, for all $n \in \mathbb{N}$.

Let $f(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_0$ be any polynomial.

Then
$$\overline{f(T)}(v+W) = f(T)(v) + W.$$
$$= (a_n T^n(v) + a_{n-1} T^{n-1}(v) + \cdots + a_0 v) + W.$$
$$= a_n (T^n(v) + W) + a_{n-1}(T^{n-1}(v) + W) + \cdots + a_0(v+W)$$
$$= a_n \bar{T}^n(v+W) + a_{n-1} \bar{T}^{n-1}(v+W) + \cdots + a_0(v+W)$$
$$= f(\bar{T})(v+W).$$
$$\Rightarrow \overline{f(T)} = f(\bar{T}).$$

If T is a zero of polynomial $f(t)$. Then
$$f(T) = 0$$
$$\Rightarrow \overline{f(T)} = \bar{0} = W$$
$$\Rightarrow f(\bar{T}) = W$$
$$\Rightarrow \bar{T} \text{ is a zero of } f(t).$$

Thus T and $\bar{T}$ are zero of same polynomial and hence the minimal polynomial of $\bar{T}$ divides the minimal polynomial of T as $\dim \frac{V}{W} < \dim V$.

<u>Theorem</u>: Let $T: V \to V$ be a linear operator whose characteristic polynomial factors into linear polynomials. Then there exists a basis of V in which T is represented by a triangular matrix.

<u>Proof</u>: Let $\dim V = n$.

We prove the result by induction on n.

If $\dim V = 1$, then the matrix representation of T is a 1×1 matrix which is triangular.

Now, we suppose that the result is true for all vector spaces whose dimension is less than n.

Given that the characteristic polynomial of T factors into linear polynomials. So T has at least one eigenvalue, say a_{11}.

Hence there exists a non-zero eigenvector v s.t. $T(v) = a_{11}v$.

Let W be a subspace spanned by v.

So $\dim W = 1$ and W is invariant under T.

Let $\bar{V} = \frac{V}{W}$. Then T induces a linear operator $\bar{T}$ on $\bar{V}$ defined by $\bar{T}(v+W) = T(v) + W$, $v \in V$, whose minimal polynomial divides the minimal polynomial of T (by previous result).

Since the characteristic polynomial of T is a product of linear polynomials. So it also its minimal polynomial. Hence the minimal and characteristic polynomial of $\bar{T}$ are same.

So characteristic polynomial of $\bar{T}$ is the product of linear factors. Hence by induction, there exists a basis $\{\bar{v}_2, \bar{v}_3, \cdots, \bar{v}_n\}$ of $\bar{V}$, where $\bar{v}_j = v_j + W$, $v_j \in V$. such that

$$\bar{T}(\bar{v}_2) = a_{22}\bar{v}_2$$
$$\bar{T}(\bar{v}_3) = a_{23}\bar{v}_2 + a_{33}\bar{v}_3$$
$$\cdots\cdots$$
$$\bar{T}(\bar{v}_n) = a_{2n}\bar{v}_2 + a_{3n}\bar{v}_3 + \cdots + a_{nn}\bar{v}_n$$

Now, since $v_2, v_3, \cdots, v_n \in V$. So the set $\beta = \{v, v_2, v_3, \cdots, v_n\}$ is a basis of V.

Since $\bar{T}(\bar{v}_2) = a_{22}\bar{v}_2 \Rightarrow T(v_2) + W = a_{22}(v+W) = a_{22}v + W$

$$\Rightarrow T(v_2) - a_{22}v \in W$$
$$\Rightarrow T(v_2) - a_{22}v = a_{12}v$$
$$\Rightarrow T(v_2) = a_{12}v + a_{22}v.$$

Similarly, $T(v_3) = a_{13}v + a_{23}v_2 + a_{33}v_3$

$$\cdots\cdots$$

$$T(v_n) = a_{1n}v + a_{2n}v_2 + \cdots + a_{nn}v_n.$$

Then $[T]_\beta = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ 0 & & & a_{nn} \end{bmatrix}$, which is triangular.

Corollary: Let A be a square matrix whose characteristic polynomial factors into linear polynomials. Then A is similar to a triangular matrix i.e. there exists an invertible matrix P such that $P^{-1}AP$ is triangular.

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$$= a_n (T^n(v) + W) + a_{n-1}(T^{n-1}(v) + W) + \cdots + a_0(v+W)$$
$$= a_n \overline{T^n}(v+W) + a_{n-1} \overline{T^{n-1}}(v+W) + \cdots + a_0(v+W)$$
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