## Nilpotent Operators

A linear operator $T: V \to V$ is said to be nilpotent if $T^k = 0$ for some +ve integer $k$. The least positive $k$ such that $T^k = 0$ is known as index of nilpotency of $T$.

Moreover, a square matrix is said to be nilpotent if $A^k = 0$ for some +ve integer $k$. The least positive $k$ such that $A^k = 0$ is known as index of $A$.

So $A$ annihilates $\lambda^k = 0$ for least $k$

minimal polynomial $= t^k$, where $k$ is the index of nilpotency

characteristic polynomial $= t^n$, $\dim V = n$.

Clearly $0$ is the only eigenvalue of $A$.

$\therefore$ trace of a nilpotent matrix is $0$.

The matrix $N = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{k \times k}$ is called Jordan Nilpotent block.

Jordan block matrix is written as

$J(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ - & - & - & - & - & - \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}$

① $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow A^2 = 0$.
Index of $A = 2$

② $A = \begin{bmatrix} 0 & 2 & 1 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow A^4 = 0$
& $A^3 \neq 0, A^2 \neq 0$.

Index of $A = 4$.

$\Rightarrow$ $J(\lambda) = \lambda I + N$.

Ex. $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} \Rightarrow A^3 = 0 \Rightarrow A^4 = A^5 = \cdots = 0$

$\therefore$ Index of matrix $A = 0$.

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<u>Theorem 1</u>: Let $T: V(F) \to V(F)$ be a linear operator. If for $v \in V$, $T^k(v) = 0$ but $T^{k-1}(v) \neq 0$. Prove that

(i) The set $S = \{v, T(v), \ldots, T^{k-1}(v)\}$ is linearly indepent.

(ii) The subspace $W$ generated by $S$ is T-invariant

(iii) The restriction $\tilde{T}$ of $T$ on $W$ is nilpotent of index $k$.

(iv) Relative to the basis $\{T^{k-1}(v), \ldots, T(v), v\}$ of $W$, the matrix of $T$ is the $k$-square Jordan nilpotent block $N_i$ of index $k$.

<u>Proof</u>: (i) Suppose $\alpha, \alpha_1, \alpha_2, \ldots, \alpha_{k-1} \in F$ such that

$$\alpha v + \alpha_1 T(v) + \alpha_2 T^2(v) + \cdots + \alpha_{k-1} T^{k-1}(v) = 0 \quad \text{---} \ (1)$$

Operate both side of (1) by $T^{k-1}$, we get

$$\alpha T^{k-1}(v) = 0 \qquad \{\because T^k(v) = 0 \Rightarrow T^{k+1}(v) = 0, \ldots\}$$

$$\Rightarrow \alpha = 0 \qquad \{\because T^{k-1}(v) \neq 0\}$$

Again operate both side of (1) by $T^{k-2}$, we get

$$\alpha T^{k-2}(v) + \alpha_1 T^{k-1}(v) = 0$$

$$\Rightarrow \alpha_1 T^{k-1}(v) = 0 \qquad \{\because \alpha = 0\}$$

$$\Rightarrow \alpha_1 = 0 \qquad \{\because T^{k-1}(v) \neq 0\}$$

Continuing the process, we get

$$\alpha = \alpha_1 = \alpha_2 = \cdots = \alpha_{k-1} = 0.$$

Thus $S$ is linearly independent.

(ii) Given $W = L(S)$

Let $w \in W$. Then $w = \beta v + \beta_1 T(v) + \cdots + \beta_{k-1} T^{k-1}(v)$

$$\Rightarrow T(w) = \beta T(v) + \beta_1 T^2(v) + \cdots + \beta_{k-2} T^{k-1}(v) + \beta_{k-1} T^k(v)$$

$$\Rightarrow T(w) = \beta T(v) + \beta_1 T^2(v) + \cdots + \beta_{k-2} T^{k-1}(v) \quad \{\because T^k(v) = 0\}$$

$$\therefore T(w) \in W.$$

Thus $W$ is T-invariant.

(iii) Since $\tilde{T}$ is the restriction of $T$ on $W$ so.

$$\tilde{T}(\omega) = T(\omega) \quad \text{for all } \omega \in W,$$

$$\Rightarrow \tilde{T}(\tilde{T}(\omega)) = \tilde{T}(T(\omega)) \quad [\because W \text{ in } T\text{-invariant So } T(\omega) \in W]$$

$$\Rightarrow \tilde{T}^2(\omega) = T^2(\omega)$$

In the similar way, $\tilde{T}^k(\omega) = T^k(\omega)$, $\forall \omega \in W$

Since $\{v, T(v), \ldots, T^{k-1}(v)\}$ generates $W$.

So for $i = 0, 1, 2, \ldots k-1$

$$\tilde{T}^k(T^i(v)) = T^{k+i}(v) = 0$$

$$\Rightarrow \tilde{T}^k = 0$$

$\Rightarrow$ Index of $\tilde{T} \geq k$.

But $\tilde{T}^{k-1}(v) = T^{k-1}(v) \neq 0$

$\therefore$ Index of $\tilde{T} = k$.

i.e. Nilpotency of $\tilde{T} = k$.

(iv) Since $\{T^{k-1}(v), T^{k-2}(v), \ldots, T(v), v\}$ is a basis of $W$.

Then $\tilde{T}(T^{k-1}(v)) = T^k(v) = 0$

$$\tilde{T}(T^{k-2}(v)) = T^{k-1}(v) = 1.T^{k-1}(v) + 0.T^{k-2}(v) + \cdots + 0.v$$

$$\tilde{T}(T^{k-3}(v)) = T^{k-2}(v) = 0.T^{k-1}(v) + 1.T^{k-2}(v) + \cdots + 0.v$$

$$\cdots\cdots\cdots$$

$$\tilde{T}(T(v)) = T^2(v) = 0.T^{k-1}(v) + 0.T^{k-2}(v) + \cdots + 1.T^2(v) + 0\,T(v) + 0.v$$

$$\tilde{T}(v) = T(v) = 0.T^{k-1}(v) + \cdots + 1.T(v) + 0.v.$$

$$\therefore [T]_S = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdots & & & & & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} = N_k.$$

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**Result.** Let $T: V \to V$ be linear and $X = \operatorname{Ker} T^{i-2}$, $Y = \operatorname{Ker} T^{i-1}$ and $Z = \operatorname{Ker} T^{i}$. Then

(i) $X \subseteq Y \subseteq Z$.

(ii) Suppose $\{u_1, \dots, u_r\}$, $\{u_1, \dots, u_r, v_1, \dots, v_s\}$, $\{u_1, u_2, \dots, u_r, v_1, \dots, v_s, w_1, \dots, w_t\}$ are bases of $X, Y, Z$ respectively. Show that $S = \{u_1, \dots, u_r, T(w_1), \dots, T(w_t)\}$ is contained in $Y$ and is linearly independent.

**Solution:** (i) Let $x \in X = \operatorname{Ker} T^{i-2} \Rightarrow T^{i-2}(x) = 0 \Rightarrow T^{i-1}(x) = T(0) = 0$
$\Rightarrow x \in \operatorname{Ker} T^{i-1} = Y$

$\therefore$ $X \subseteq Y$. Similarly $Y \subseteq Z$. So $X \subseteq Y \subseteq Z$.

(ii) Let $z \in Z = \operatorname{Ker} T^{i} \Rightarrow T^{i}(z) = 0 \Rightarrow T^{i-1}(T(z)) = 0 \Rightarrow T(z) \in \operatorname{Ker} T^{i-1}$
$\Rightarrow T(Z) \subseteq Y$.
$\Rightarrow S \subseteq Y$.

To show $S$ is linearly independent, we suppose $S$ is linearly dependent. So for $\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_t \in F$ (not all zero) s.t.

$$\alpha_1 u_1 + \dots + \alpha_r u_r + \beta_1 T(w_1) + \dots + \beta_t T(w_t) = 0.$$

If $\beta_i = 0 \ \forall i$ Then $\alpha_i = 0 \ \forall i$ $\{\because u_1, \dots, u_r$ are l.i.$\}$

So at least one $\beta_i \neq 0$

Now $\beta_1 T(w_1) + \dots + \beta_t T(w_t) = (-\alpha_1) u_1 + \dots + (-\alpha_r) u_r \in X = \operatorname{Ker} T^{i-2}$

$\Rightarrow T^{i-2}(\beta_1 T(w_1) + \dots + \beta_t T(w_t)) = 0$

$\Rightarrow T^{i-1}(\beta_1 w_1 + \dots + \beta_t w_t) = 0 \Rightarrow \beta_1 w_1 + \dots + \beta_t w_t \in \operatorname{Ker} T^{i-1} = Y$.

$\Rightarrow \beta_1 w_1 + \dots + \beta_t w_t = \gamma_1 u_1 + \dots + \gamma_r u_r + \delta_1 v_1 + \dots + \delta_s v_s$ $\{\{u_1, \dots, u_r, v_1, \dots, v_s\}$ is a basis of $Y\}$

$\Rightarrow \gamma_1 u_1 + \dots + \gamma_r u_r + \delta_1 v_1 + \dots + \delta_s v_s + (-\beta_1) w_1 + \dots + (-\beta_t) w_t = 0$

$\Rightarrow \gamma_i = 0, \ \delta_j = 0, \ \beta_k = 0 \ \forall i, j, k$.

Thus we get a contradiction because $\beta_i \neq 0$, for at least one $i$

So $S$ is linearly independent.

**Theorem 2:** Let $T: V \to V$ be a nilpotent operator of index $k$. Then $T$ has a unique block diagonal matrix representation consisting of Jordan nilpotent blocks $N$. There is at least one $N$ of order $k$ and all other are of orders $\leq k$. The total number of $N$ of all orders is equal to the nullity of $T$.

**Proof:** Let $V$ be $n$-dimensional vector space and $T: V \to V$ be a nilpotent operator of index $k$.

Let $W_i = \ker T^i$, $i = 1, 2, \ldots, k$.

and $\dim W_i = m_i$.

Suppose $v \in V$ then $T^k(v) = 0$

$$\Rightarrow v \in \ker T^k = W_k$$

$$\therefore V \subseteq W_k$$

$$\Rightarrow W_k = V.$$

Also $W_{k-1} \neq V$.

$$\therefore m_{k-1} < m_k = n.$$

By previous result, $W_1 \subseteq W_2 \subseteq \cdots \subseteq W_k = V$ and if $\{u_1, u_2, \ldots u_n\}$ is a basis of $V$, then $\{u_1, u_2, \ldots, u_{m_i}\}$ is a basis of $W_i$.

Choose a new basis of $V$ for which we get the desired matrix representation of $T$ as,

$v(1,k) = u_{m_{k-1}+1}$, $v(2,k) = u_{m_{k-1}+2}, \ldots, v(m_k - m_{k-1}, k) = u_{m_k}$.

and setting,

$v(1, k-1) = Tv(1,k)$, $v(2,k-1) = Tv(2,k), \ldots\ldots$

$v(m_k - m_{k-1}, k-1) = Tv(m_k - m_{k-1}, k)$

Again by previous result,

$$S_1 = \{u_1, \ldots u_{m_{k-2}}, v(1, k-1), \ldots, v(m_k - m_{k-1}, k-1)\}$$

is a linearly independent subset of $W_{k-1}$.
By extension theorem on basis, $S_1$ canbe extend to form a basis of $W_{k-1}$ by inserting new elements,

$$v(m_k - m_{k-1} + 1, k-1), \ldots\ldots, v(m_{k-1} - m_{k-2}, k-1).$$

Next, we set,

$$v(1, k-2) = Tv(1, k-1),\ v(2, k-2) = T(v(2, k-1)) \ldots$$
$$v(m_{k-1} - m_{k-2}, k-2) = Tv(m_{k-1} - m_{k-2}, k-1)$$

Again by previous result,

$$S_2 = \{u_1, \ldots, u_{k-3}, v(1, k-2), \ldots, v(m_{k-1} - m_{k-2}, k-2)\}$$

is a linearly independent subset of $W_{k-2}$ and it extends to form a basis of $W_{k-2}$ by inserting elements,

$$v(m_{k-1} - m_{k-2} + 1, k-2), v(m_{k-1} - m_{k-2} + 2, k-2), \ldots,$$
$$v(m_{k-2} - m_{k-3}, k-2),$$

Continuing in the similar way, we get a new basis for $V$, which canbe arranged as follows,

$$v(1, k), \ldots, v(m_k - m_{k-1}, k)$$
$$v(1, k-1), \ldots, v(m_k - m_{k-1}, k-1), \ldots\ldots v(m_{k-1} - m_{k-2}, k-1)$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$v(1, 2) \ldots, v(m_k - m_{k-1}, 2), \ldots v(m_{k-1} - m_{k-2}, 2), \ldots v(m_2 - m_1, 2)$$
$$v(1, 1), \ldots, v(m_k - m_{k-1}, 1), \ldots v(m_{k-1} - m_{k-2}, 1) \ldots v(m_2 - m_1, 1), \ldots v(m_1, 1)$$

The ~~last~~ row forms a basis of $W_1$, the Last two row forms a basis of $W_2$ and so on

This implies that

$$Tv(i,j) = \begin{cases} v(i,j-1) & , \text{ for } j>1 \\ 0 & , \text{ for } j=1 \end{cases}$$

It is clear from Theorem 1 (N) that T have the desired form if the $v(i,j)$ are ordered lexicographically, begins with $v(1,1)$ ~~and~~ to $v(1,k)$ in the first column then moving from $v(2,1)$ to $v(2,k)$ in the 2nd column and so on.

There will be exactly $m_k - m_{k-1}$ diagonal entries of order k.

Also, $(m_{k-1}-m_{k-2})-(m_k-m_{k-1}) = 2m_{k-1}-m_k-m_{k-2}$ diagonal entries of order $k-1$

.........

$2m_2-m_1-m_3$ diagonal entries of order 2

$2m_1-m_2$ diagonal entries of order 1.

We know that $m_i = \dim \ker T^i$ is uniquely determined by T; so the number of diagonal entries of each order is uniquely determined by T.

Since $m_1 = (m_k - m_{k-1}) + (2m_{k-1}-m_k-m_{k-2}) + \cdots + (2m_2-m_1-m_3) + (2m_1-m_2)$

$\Rightarrow m_1 =$ Nullity T is the total number of diagonal block of T.

Ex: $A = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow A^3 = 0$  Index of $A = 3$

Rank $A = 2$

$\therefore$ Nullity $A = 5-2 = 3$.

So one Jordan Nilpotent block is of order 3 and other blocks are of order 1.

<u>Note</u>: Two matrices of same nilpotency have different number of Jordan nilpotent block and hence are not similar.