Vibration Of Finite String With Fixed Ends

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Suppose that vibration of the string of length L fixed at both sides (ends).

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},\ 0 < x < L,\ t > 0 \qquad (1)$$

$$u(x, 0) = f(x), \qquad 0 \leq x \leq L \qquad (2)$$

$$u_t(x, 0) = g(x), \qquad 0 \leq x \leq L \qquad (3)$$

$$u(0, t) = 0,\ u(L, t) = 0,\ t \geq 0 \qquad (4)$$

We know that the solution of the wave equation $u(x,t) = \emptyset(x+ct) + \varphi(x-ct)$

Use the initial conditions, we obtain

$$u(x,0) = \emptyset(x) + \varphi(x) = f(x), \qquad 0 \le x \le L \tag{5}$$

$$u_t(x,0) = c\,[\emptyset'(x) - \varphi'(x)] = g(x), \qquad 0 \le x \le L \tag{6}$$

Solving from equation (5) and (6), we get the value of $\emptyset(v)$ and $\varphi(w)$

$$\emptyset(v) = \frac{1}{2} f(v) + \frac{1}{2c}\int_0^v g(\alpha)d\alpha + \frac{k}{2}, \qquad 0 \le v \le L \tag{7}$$

$$\varphi(w) = \frac{1}{2} f(w) - \frac{1}{2c}\int_0^w g(\alpha)d\alpha - \frac{k}{2}, \qquad 0 \le w \le L \tag{8}$$

There fore, the required solution

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\alpha)d\alpha,$$

for, $0 \leq x + ct \leq L$ and $0 \leq x - ct \leq L$. The solution is determined by the initial data $t \leq \frac{x}{c}$, and $t \leq \frac{L-x}{c}$, $t \geq 0$.

For larger times, applying the boundary conditions, we obtain

$$u(0,t) = \emptyset(ct) + \varphi(-ct) = 0, \qquad t \geq 0, \qquad (9)$$

$$u(L,t) = \emptyset(L+ct) + \varphi(L-ct) = 0, \qquad t \geq 0, \quad (10)$$

Setting $\beta = -ct$, equation (9) reduces to

$$\varphi(\beta) = -\emptyset(-\beta), \qquad \beta \leq 0, \qquad (11)$$

Setting $\beta = L + ct$, equation (10) takes the form

$$\emptyset(\beta) = -\varphi(2L - \beta), \quad \beta \geq L, \qquad (12)$$

If we take $v = -w$, then rewrite the equation (7) as

$$\emptyset(-w) = \frac{1}{2} f(-w) + \frac{1}{2c}\int_0^{-w} g(\alpha)d\alpha + \frac{k}{2}, \quad 0 \leq -w \leq L \qquad (13)$$

Now, from equation (11) and (13), we obtain

$$\varphi(w) = -\frac{1}{2} f(-w) - \frac{1}{2c}\int_0^{-w} g(\alpha)d\alpha - \frac{k}{2}, \quad -L \leq w \leq 0 \qquad (14)$$

We observe that the range of $\varphi(w)$ is extended to $-L \leq w \leq L$

If we taken $\beta = v$ then equation (12) reduces to

$$\emptyset(v) = -\varphi(2L - v), \quad v \geq L, \qquad (15)$$

Putting $w = 2L - v$ in equation (8), we get

$$\varphi(2L - v) = \frac{1}{2} f(2L - v) - \frac{1}{2c}\int_0^{2L-v} g(\alpha)d\alpha - \frac{k}{2}, \quad 0 \leq 2L - v \leq L \qquad (16)$$

Now from equation (15) and (16), we obtain

$$\emptyset(v) = -\frac{1}{2} f(2L - v) + \frac{1}{2c}\int_0^{2L-v} g(\alpha)d\alpha + \frac{k}{2}, \quad L \leq v \leq 2L \qquad (17)$$

In above expression, the range of $\emptyset(v)$ is extended to the interval

$L \leq v \leq 2L$. Therefore, we obtain $\emptyset(v),\ \forall\, v \geq 0$ and $\varphi(w)$,

$\forall\, w \leq L$. The solution is determined $\forall\, 0 \leq x \leq L$ and $t \geq 0$.

Question 1: <u>Determine the solution of the following problem</u>

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L\,, \quad t > 0,$$

$$u(x,0) = \sin\left(\frac{\pi x}{L}\right), 0 \leq x \leq L$$

$$u_t(x,0) = 0, \ \ 0 \leq x \leq L$$

$$u(0,t) = 0, \qquad u(L,t) = 0, \qquad t \geq 0$$

Question 2: Determine the solution of the following problem

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L\,, \quad t > 0,$$

$$u(x,0) = f(x)\; 0 \le x \le L$$

$$u_t(x,0) = g(x), \;\; 0 \le x \le L$$

$$u(0,t) = p(t), u(L,t) = q(t), t \ge 0$$

Thank you