# DUALITY IN LINEAR PROGRAMMING

By

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# DUALITY IN LINEAR PROGRAMMING PROBLEM

There is always a corresponding Linear Programming Problem (LPP) associated to every LPP, which is called as dual problem of the original LPP(Primal Problem).

Let the **primal problem** in standard form be

max/min $Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$

$s.t.$ $a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$

.

.

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$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m$

and $x_1, x_2, \ldots, x_n \geq 0$

**Associated dual problem is**

$$\min/max\ \ Z^* = b_1 w_1 + b_2 w_2 + \ldots + b_m w_m$$
$$s.t.\ a_{11} w_1 + a_{21} w_2 + \ldots + a_{m1} w_m \geq / \leq c_1$$
$$a_{12} w_1 + a_{22} w_2 + \ldots + a_{m2} w_m \geq / \leq c_2$$
.
.
.
$$a_{1n} w_1 + a_{2n} w_2 + \ldots + a_{mn} w_m \geq / \leq c_n$$
and $w_1, w_2, \ldots, w_m$ are unrestricted in sign.

# FORMULATING A DUAL PROBLEM

Steps for formulation are summarised as

**Step 1:** write the given LPP in its standard form.

**Step 2:** identify the variables of dual problem which are same as the number of constraints equation.

**Step 3:** write the objective function of the dual problem by using the constants of the right had side of the constraints.

**Step 4:** if primal is max/min type than dual is min/max type and the constraints are $\geq / \leq$ type.

**Step 5:** dual variables are unrestricted in sign.

# PRIMAL-DUAL PAIR IN MATRIX FORM

**Standard Primal Problem:**

Find $x \in \mathbb{R}^n$ so as to

maximize/minimize z $=$ c$x, c \in \mathbb{R}^n$

Subject to constraints

$Ax = b \ and \ x \geq 0, b \in \mathbb{R}^m$

where $A \ is \ m \times n$ real matrix.

**Associated Dual Problem:**

Find w$\in \mathbb{R}^m$ so as to

minimize/maximize $z^* = b^T w, b \in \mathbb{R}^m$

Subject to constraints

$A^T w \geq / \leq c^T,$ c $\in \mathbb{R}^n$

where $A \ is \ n \times m$ real matrix and $w$ is unrestricted in sign.

# SOME IMPORTANT CHARACTERISTICS OF DUALITY

- Dual of dual is primal.
- If either the primal or dual problem has a solution then the other also has a solution and their optimum values are equal.
- If any of the two problems has an infeasible solution, then the value of the objective function of the other is unbounded.
- The value of the objective function for any feasible solution of the primal is less than the value of the objective function for any feasible solution of the dual.

- If either the primal or dual has an unbounded solution, then the solution to the other problem is infeasible.

- If the primal has a feasible solution, but the dual does not have then the primal will not have a finite optimum solution and vice versa.

# DUALITY AND SIMPLEX METHOD

Since any LPP can be solved by using simplex method, so we can solve primal as well as dual, and as there is relation between the two we can solve one and give the solution for the associated other problem by using some rule as given

Primal problem is in form of maximization, then

***Rule 1:*** Corresponding net evaluation of the starting primal variables=difference between the left and right sides of the dual constraints associated with the starting primal variables.

(by using optimal solution of the primal get the optimal solution for dual.)

***Rule 2:*** negative of the Corresponding net evaluation of the starting dual variables=difference between the left and right sides of the primal constraints associated with the starting dual variables.
(by using optimal solution of the dual get the optimal solution for primal.)

***Rule 3:*** if the primal (dual) is unbounded then the dual (primal) does not have any feasible solution.

# ADVANTAGES AND APPLICATIONS OF DUALITY

- Sometimes dual problem solution may be easier than primal solution, particularly when the number of decision variables is considerably less than slack / surplus variables.
- In the areas like economics, it is highly helpful in obtaining future decision in the activities being programmed.
- In physics, it is used in parallel circuit and series circuit theory.

- In game theory, dual is employed by column player who wishes to minimize his maximum loss while his opponent i.e. Row player applies primal to maximize his minimum gains. However, if one problem is solved, the solution for other also can be obtained from the simplex tableau.
- When a problem does not yield any solution in primal, it can be verified with dual.
- Economic interpretations can be made and shadow prices can be determined enabling the managers to take further decisions.

## SOME EXAMPLES

Write the dual of following LPP

**1.** $minimize\ z = 20x_1 + 40x_2$

$s.t.\ 36\,x_1 + 6x_2 \geq 108,$

$3x_1 + 12x_2 \geq 36,$

$20\,x_1 + 10x_2 \geq 100\ and$

$x_1 \geq 0, x_2 \geq 0$

**Solution:** first we have to write the given LPP in standard form.

introducing surplus variables $s_1, s_2, s_3 \geq 0$, the primal problem in standard form can be written as

$$minimize\ z = 20x_1 + 40x_2 + 0.s_1 + 0.s_2 + 0.s_3$$

$$s.t.\ 36\,x_1 + 6x_2 - s_1 = 108,$$

$$3x_1 + 12x_2 - s_2 = 36,$$

$$20\,x_1 + 10x_2 - s_3 = 100\ and$$

$$x_1, x_2, s_1, s_2, s_3 \geq 0.$$

**Associated dual is**

$$maximize\ z^* = 108w_1 + 36w_2 + 100w_3$$

$$s.t.\ 36w_1 + 3w_2 + 20w_3 \leq 20,$$

$$6w_1 + 12w_2 + 10w_3 \leq 40,$$

$$-w_1 \leq 0,\ -w_2 \leq 0, -w_3 \leq 0$$

Where $w_1, w_2, w_3$ are unrestricted in sign

dominated by $w_1 \geq 0$, $w_2 \geq 0, w_3 \geq 0$

**2**. $maximize\ z = x_1 + 3x_2$

$s.t.\ \ x_1 + x_2 \leq 3,$

$2x_1 - x_2 \geq -1,$

$x_1 + 2x_2 = 5, and$

$x_1 \geq 0, x_2$ unrestricted in sign.

**Solution:** first we have to write the given LPP in standard form.

let $x_2 = x_2' - x_2''$ (where $x_2'\ and\ x_2'' \geq 0$)
introducing slack variables $s_1, s_2 \geq 0$,
the primal problem in standard form can be written as

$$maximize\ z = x_1 + 3x_2' - 3x_2'' + 0.s_1 + 0.s_2$$

$$s.t.\ \ x_1 + x_2' - x_2'' + s_1 = 3,$$

$$-2x_1 + x_2' - x_2'' + s_2 = 1,$$

$$x_1 + 2x_2' - 2x_2'' = 5, and$$

$$x_1, x_2', x_2'', s_1, s_2 \geq 0.$$

**Associated dual is**

$$minimize\ z^* = 3w_1 + w_2 + 5w_3$$

$$s.t.\ w_1 - 2w_2 + w_3 \geq 1,$$

$$\left.\begin{array}{r} w_1 + w_2 + 2w_3 \geq 3, \\ -w_1 - w_2 - 2w_3 \geq -3, \end{array}\right\} \Rightarrow w_1 + w_2 + 2w_3 = 3$$

$$w_1 \geq 0, w_2 \geq 0, w_3 \text{ unrestricted.}$$

# Thank You!