Jordan Canonical form.

Recall, Block diagonal matrix is written as

$$A = \begin{bmatrix} A_{11} & \cdots & 0 \\ & A_{22} & \\ 0 & & \ddots \\ & & & A_{kk} \end{bmatrix}$$

where order of A = Sum of order of A_{ii}.

$$Ch_A(x) = \prod Ch_{A_{ii}}(x)$$

$$= Ch_{A_{11}}(x)\, Ch_{A_{22}}(x) \cdots Ch_{A_{kk}}(x).$$

$$m_A(x) = \operatorname{lcm}\{ m_{A_{ii}}(x) ;\ i = 1, 2, \dots, k\}$$

Ex: ① $A = \begin{bmatrix} 2 & 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$, Here $A_{11} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$A_{22} = \begin{bmatrix} 1 & 5 \\ 0 & 3 \end{bmatrix}$

$A_{33} = [4]$.

$$Ch_{A_{11}}(x) = (x-2)^2 = m_{A_{11}}(x)$$

$$Ch_{A_{22}}(x) = (x-1)(x-3) = m_{A_{22}}(x)$$

$$Ch_{A_{33}}(x) = x-4 = m_{A_{33}}(x).$$

$$\therefore Ch_A(x) = (x-2)^2(x-1)(x-3)(x-4) = m_A(x)$$

② $A = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$, Here $A_{11} = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ $A_{22} = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$.

$$Ch_{A_{11}}(x) = (x-3)^2 = m_{A_{11}}(x)$$

$$Ch_{A_{22}}(x) = (x-3)(x-5) = m_{A_{22}}(x).$$

$$\therefore Ch_A(x) = (x-3)^3(x-5)$$

$$m_A(x) = (x-3)^2(x-5).$$

Jordan Block matrix:

A matrix A of the form $\begin{bmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix}$ is known as Jordan Block.

Jordan Canonical form of a matrix:-

A matrix is said to be in Jordan Canonical (J.C.) form if it is a block diagonal matrix and each block is Jordan blocks.

Example:

① Eigenvalues are 3, 3, 2, then J.C. form will be

$$\begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

② Eigenvalues are 3, 3, 3, then J.C. form will be

$$\begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}, \quad \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

③ Eigenvalues are 3, 3, 4, 4; then J.C. form will be

$$\begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \quad \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \quad \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \quad \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

<u>Theorem</u>: Let $T: V \to V$ be a linear operator whose characteristic and minimal polynomials are respectively.

$$ch(t) = (t-\lambda_1)^{n_1} \cdots (t-\lambda_r)^{n_r} \quad \text{and}$$
$$m(t) = (t-\lambda_1)^{m_1} \cdots (t-\lambda_r)^{m_r}$$

where the λ_i are distinct scalars. Then T has a block diagonal matrix representation J in which each diagonal entry is a Jordan block $J_{ij} = J(\lambda_i)$. For each λ_i, the corresponding J_{ij} have the following properties:

(i) There is at least one J_{ij} of order m_i, all other J_{ij} are of order less than m_i.

(ii) The sum of order J_{ij} is n_i.

(iii) The number of J_{ij} equals the geometric multiplicity of λ_i.

(iv) The number of J_{ij} of each possible order is uniquely determined by T.

- The proof of this theorem is the consequence of the theorem on 'Nilpotent operator' by using primary decomposition theorem.

<u>Result</u>: Two matrices are similar iff they have the same J.C. forms upto order of Jordan block.

From the theorem:

Number of Jordan block = Geometric multiplicity
= No of linearly Independent vectors.
= Nullity $(J - \lambda_i I)$

= $n -$ Rank $(A - \lambda_i I) = k$.

$\therefore$ Number of Jordan block corresponding to λ_i is equal to $\dim (J - \lambda_i I)$

Example:

$$J = \left[\begin{array}{cc|cc|c} 4 & 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ \hline 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ \hline 0 & 0 & 0 & 0 & 2 \end{array}\right] = \begin{bmatrix} J_2(4) & & \\ & J_2(2) & \\ & & J_1(2) \end{bmatrix}$$

Here eigenvalues of J are 4 and 2 with algebraic multiplicity 2 and 3 respectively.

Case(I) $\lambda_1 = 4$

Nullity of $(J - 4I) = 5 - \text{Rank}(J - 4I) = 5 - 4 = 1$.

Here 1 linearly Independent vectors corresponding to $\lambda = 4$

So No of Jordan block corresponding to $\lambda = 4$ is 1.

Case(II) $\lambda = 2$

Nullity $(J - 2I) = 5 - \text{Rank}(J - 2I) = 5 - 3 = 2$.

Here 2 linearly independent vectors corresponding to $\lambda = 2$.

Thus it clear that J.C. Form have only one J.block for eigenvalues 4 and two J.block for eigenvalue 2

Here algebraic multiplicity of eigenvalues 4 and 2 are 2 and 3 respectively where as geometric multiplicity are 1 and 2.

How to find J.C. form of a matrix

Let $A = [a_{ij}]_{n \times n}$ be a square matrix

(1) Find $Ch_A(x)$ and eigenvalues of A as

$$Ch_A(x) = \prod_{i=1}^{k} (x-\lambda_i)^{n_i} \quad \text{and} \quad \sum n_i = n.$$

(2) Find minimal polynomial of A, say

$$m_A(x) = \prod_{i=1}^{k} (x-\lambda_i)^{m_i}, \quad \sum m_i \le n.$$

(3) In the J.C. form, for each λ_i, there is at least one block of order m_i and other block corresponding to same λ_i are of order less than or equal to m_i.

(4) Sum of order of all blocks corresponding to λ_i is equal to n.

$$\sum o(A_{ii}) = o(A) = n.$$

(5) The number of Jordan blocks corresponding to λ_i is equal to G.M. of λ_i, which is equal to l.i. vectors.

(6) The number of blocks of each possible order is uniquely determined by T.

Example: (1) Write J.C. form if

$$Ch_A(x) = (x-3)^3 (x+7)^2$$
$$m_A(x) = (x-3)^2 (x+7)$$

(i) At least one block of order 2 corresponding to $\lambda_1 = 3$ other block are order less than or equal to 2.

(ii) At least one block of order 1 corresponding to $\lambda = 7$. So there must be two blocks corresponding to $\lambda = 7$ of order 1,

Thus
$$J = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 7 \end{bmatrix}.$$

<u>Example ②</u> Suppose, $Ch_A(x) = (x-4)^5$

$m_A(x) = (x-4)^2$.

So there is at least one J. block of order 2 corresponding to eigenvalues $\lambda = 4$. So

$$\begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}, \quad \begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}.$$

We get 2 J.C. form, so have different <u>G.M.</u>

Thus if $Ch_A(x)$ & $m_A(x)$ are given for an unknown matrix A. Then J.C. form may not be unique but have always finite number of possibilities of J.C forms.

* If $Ch_A(x) = \prod_{i=1}^{k} (x-\lambda_i)^{n_i}$, Then number of possible J.C form $= \prod_{i=1}^{k} p(n_i) = p(n_1)\,p(n_2) \cdots p(n_k)$

where ~~for~~ $p(n_i)$ is the no of partition of n_i.

<u>Ex</u>: Suppose $Ch_A(x) = (x-2)^5$. Then possible minimal polynomials are $(x-2), (x-2)^2, (x-2)^3, (x-2)^4, (x-2)^5$.

If $m_A(x) = x-2$. Then only 1 J.C form. $\begin{bmatrix} 2 & & 0 \\ & \ddots & \\ 0 & & 2 \end{bmatrix}$

If $m_A(x) = (x-2)^2$. Then 2 J.C. form as examples before.

If $m_A(x) = (x-2)^3$. Then 2 J.C. form are possible like

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

If $m_A(x) = (x-2)^4$. Only 1 J.C form are possible

If $m_A(x) = (x-2)^5$. Only 1 J.C form are possible.

Thus there are $7 = p(5)$ possible J.C forms.

Scanned with CamScanner

Ex. Suppose $Ch_A(x) = (x-3)^3 (x-4)^2$

$n_1 = 3, \ n_2 = 2$

The no of possible J.C. form $= p(n_1)\,p(n_2)$

$= 3 \times 2 = 6$.

Result. If $Ch_A(x) = \prod_{i=1}^{k} (x-\lambda_i)^{n_i}$

$m_A(x) = \prod_{i=1}^{k} (x-\lambda_i)^{m_i}$

Let $q_i = n_i - m_i$

Then number of possible J.C. form $= \cancel{\Pi q_i}\ \Pi s(q_i)$

where $s(q_i)$ denotes the number of partition of q_i in which no part is greater than m_i.

Ex: If $Ch_A(x) = (x-2)^7 \quad m_A(x) = (x-2)^2$

Here $n_i = 7, \ m_i = 2$

$\therefore \ q_i = n_i - m_i = 5$.

Now partition of 5 are $5+0,\ 4+1,\ 3+2,\ 3+1+1,\ 2+2+1,\ 2+1+1+1,\ 1+1+1+1+1$.

Partition of 5 in which no part is greater than $2 = m_i$ are

$2+2+1,\ 2+1+1+1,\ 1+1+1+1+1$

$\therefore \ s(q_i) = 3$.

$\therefore$ No of J.C. form $= 3$.

Ex. If $Ch_A(x) = (x-2)^4 (x-4)^5$

$m_A(x) = (x-2)^2 (x-4)^2$

$n_1 = 4, \ m_1 = 2$

$q_1 = n_1 - m_1 = 4 - 2 = 2$

$2 = 2+0,\ 1+1$

$s(q_1) = 2$

$n_2 = 5, \ m_2 = 2$

$q_2 = 5 - 2 = 3$

$3 = \cancel{3+0},\ 2+1,\ 1+1+1$.

$s(q_2) = 2$.

No of J.C. form $= s(q_1)\,s(q_2) = 2$ [illegible] 4.

Result: If A is nilpotent matrix of index k, then J.C form have at least one block of order k and all other block will be of order less than or equal to k.

Ex: $A = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{5\times 5}$ Rank A = 2, Here 0 is the only eigenvalues of A.M. 5.

No. of l.i. vectors $= n - \text{rank}(A - 0I) = 5 - 2 = 3$ = Number of Jordan block Corresponding to 0 is 3.

Also $A^3 = 0$, Index of nilpotency = 3.

So at least one block of J.C form is of order 3

$$\therefore J = \left[\begin{array}{ccc|c|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Q. Find the J.C form of a companion matrix $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 4 & -6 & 4 \end{bmatrix}$

We know that Characteristic polynomial of A is given by

$$Ch_A(\lambda) = \lambda^4 - 4\lambda^3 + 6\lambda^2 - 4\lambda + 1$$

$$Ch_A(\lambda) = (\lambda - 1)^4 = m_A(\lambda) \quad \{\text{found earlier}\}$$

So at least one block of order 4 is possible.

Thus only one J.C. form is possible. i.e.

$$A \sim J = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Another Way: Find rank$(A - 1I)$. $\lambda = 1$ is of A.M. 4.

we get rank$(A - 1I) = 3$.

No of Jordan block $= n - \text{rank}(A - 1I) = 4 - 3 = 1$.

and $(A - I)^4 = 0$. So at least one Jordan block of order 4.