COMPLEX ANALYSIS

By
Dr. Babita Mishra
Assistant Professor,
Department of Mathematics,
School of Physical Sciences.
Mahatma Gandhi Central University,
Motihari, Bihar.

Derivatives of an analytic function

Note:- Derivatives of all orders of an analytic function are analytic.

Theorem: Let $f(z)$ be analytic in a domain D and let γ be a simple closed contour in D, taken in (+)ve sense (anticlockwise). Then, for all points z interior to γ,

$$f'(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(s)}{(s-z)^2}\,ds \quad \longrightarrow (*)$$

Proof. To prove $(*)$ we have to show that for a given $\varepsilon > 0$, there exist a $\delta > 0$ s.t.

$$\left| \frac{f(z+h)-f(z)}{h} - \frac{1}{2\pi i}\int_\gamma \frac{f(s)}{(s-z)^2}\,ds \right| < \varepsilon$$

whenever $|h| < \delta$ $\left\{ \begin{array}{l} z \in \text{interior to } \gamma \\ z+h \text{ also belongs to interior to } \gamma \text{ as } |h|<\delta \end{array} \right\}$

now by using Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(s)}{s-z}\,ds, \quad z \in \text{interior to } \gamma$$

now,

$$\frac{f(z+h)-f(z)}{h} = \frac{1}{2\pi i h}\int_\gamma \left(\frac{1}{s-z-h} - \frac{1}{s-z}\right) f(s)\,ds$$

$$= \frac{1}{2\pi i \cancel{h}}\int_\gamma \left\{\frac{\cancel{s}-\cancel{z}-\cancel{s}+\cancel{z}+\cancel{h}}{(s-z-h)(s-z)}\right\} f(s)\,ds$$

$$= \frac{1}{2\pi i}\int_\gamma \frac{f(s)\,ds}{(s-z-h)(s-z)}$$

Hence,

$$\frac{f(z+h)-f(z)}{h} - \frac{1}{2\pi i}\int_\gamma \frac{f(s)\,ds}{(s-z)^2} = \frac{1}{2\pi i}\int_\gamma \left[\frac{1}{(s-z-h)(s-z)} - \frac{1}{(s-z)^2}\right] f(s)\,ds$$

$$= \frac{1}{2\pi i}\int_\gamma \left[\frac{\cancel{s}-\cancel{z}-\cancel{s}+\cancel{z}+h}{(s-z-h)(s-z)^2}\right] f(s)\,ds$$

$$= \frac{h}{2\pi i}\int_\gamma \frac{f(s)\,ds}{(s-z-h)(s-z)^2}$$

or

$$\left|\frac{f(z+h)-f(z)}{h} - \frac{1}{2\pi i}\int_\gamma \frac{f(s)}{(s-z)^2}ds\right| = \frac{|h|}{2\pi}\left|\int_\gamma \frac{f(s)}{(s-z-h)(s-z)^2}ds\right|$$

{as $|i|=1$}

Let $d = \min\{|s-z|\}$ {z is fixed but s varies over the path γ}

as $f(s)$ is contin on $\gamma \Rightarrow |f(s)| \le M \quad \forall s$ on γ

and

$$|s-z-h| \ge \left|\underbrace{|s-z|}_{\text{min value } d} - |h|\right| = d - |h| \quad \text{and}$$

$$|s-z| \ge d$$

using the above observation we have.

$$\left|\frac{f(z+h)-f(z)}{h} - \frac{1}{2\pi i}\int_\gamma \frac{f(s)}{(s-z)^2}ds\right| \le \frac{|h|M}{\underbrace{(d-|h|)d^2}}\underbrace{\int_\gamma ds}_{L}$$

as $h \to 0$, $|h| < \delta$ and $\to 0$; L {length of γ}

Hence

$$\boxed{\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = \frac{1}{2\pi i}\int_\gamma \frac{f(s)}{(s-z)^2}ds}$$

$\Downarrow$

since z is arbitrary $\Rightarrow$ valid for all z interior to γ.

Theorem. Let $f(z)$ be analytic in a domain D and let γ be a simple closed contour in D, taken in $(+)$ve sense. Then for all points z interior to γ,

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(s)}{(s-z)^{n+1}}ds \quad \text{—} (**), \qquad n = 0, 1, 2, \ldots$$

Proof – we prove the above equality using mathematical induction

for $n=0$ {equivalent to Cauchy's integral formula}

for $n=1$ {previous theorem}

Let $(**)$ holds for $n=m$.

Let z be an arbitrary point interior to γ and $z+h \in$ interior to γ

now,

$$\frac{f^m(z+h)-f^m(z)}{h} = \frac{m!}{2\pi i h}\int_\gamma \left[\frac{1}{(s-z-h)^{m+1}} - \frac{1}{(s-z)^{m+1}}\right] f(s)\,ds$$

$$= \frac{m!}{2\pi i h}\int_\gamma \frac{1}{(s-z)^{m+1}}\left[\frac{\cancel{(s-z)^{m+1}}}{\cancel{(s-z)^{m+1}}\left[1-\frac{h}{s-z}\right]^{m+1}} - 1\right] f(s)\,ds$$

$$= \frac{m!}{2\pi i h}\int_\gamma \frac{1}{(s-z)^{m+1}}\left[\left(1-\frac{h}{s-z}\right)^{-(m+1)} - 1\right] f(s)\,ds$$

$$= \frac{m!}{2\pi i h}\int_\gamma \frac{1}{(s-z)^{m+1}}\left[\frac{h(m+1)}{s-z} + \frac{(m+1)(m+2)h^2}{2!\,(s-z)^2} + \cdots\right] f(s)\,ds$$

$\{(1-x)^{-n}$ expansion$\}$

$$= \frac{m!}{2\pi i}\int_\gamma \frac{1}{(s-z)^{m+1}}\left[\frac{(m+1)}{s-z} + \underbrace{\frac{(m+1)(m+2)h}{2!\,(s-z)^2} + \cdots}_{\text{all terms after this} \to 0 \text{ as } h\to 0}\right] f(s)\,ds$$

taking limit $h \to 0$

$$\lim_{h\to 0}\frac{f^m(z+h)-f^m(z)}{h} = \frac{m!}{2\pi i}\int_\gamma \frac{f(s)}{(s-z)^{m+1}}\,\frac{m+1}{(s-z)}\,ds$$

or $$f^{m+1}(z) = \frac{(m+1)!}{2\pi i}\int_\gamma \frac{f(s)\,ds}{(s-z)^{m+2}}$$

$\Rightarrow$ (**) also holds for $n = m+1$ if it holds for $n = m$

$\Rightarrow$ (**) holds for all n.

Remark

① from (**) we can see that if f is analytic at a point then its derivatives of all order exist in some nbd of that pt.

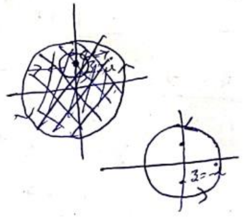
$\Rightarrow$ derivatives of all orders of an analytic fns are also analytic.

② If a fns is analytic at a pt. then its component fns u and v have contin. partial derivatives of all orders at this point.

Examples on Cauchy's Integral formula and derivatives of an analytic function.

Ex-1 Evaluate $\int_\gamma \frac{z^2-4z+4}{z+i}\,dz$ where γ is the circle $|z|=2$

Sol^n: let $f(z) = z^2-4z+4$, $z=-i$ point lies inside the circle $|z|=2$



using Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(s)}{s-z}\,ds \qquad z \notin \gamma$$

$$\int_\gamma \frac{f(s)}{s-z}\,ds = f(z)\cdot 2\pi i$$

In this particular prob. $f(s) = s^2-4s+4$ & $z=-i$

$$f(-i) = (-i)^2 - 4\times(-i)+4 = -1+4i+4 = 4i+3$$

so $$\int_\gamma \frac{z^2-4z+4}{z+i}\,dz = 2\pi i\,(4i+3) = -8\pi+6\pi i$$
$$= \pi\,(6i-8) \text{ Ans.}$$

Ex-2 Evaluate $\int_\gamma \frac{\cos z}{z^3+z}\,dz$ where γ is the circle $|z|=2$

Sol^n First we have to see the points where $f(z)$ is not analytic so we have to see denominator in factor form, again to see the prob, so that we can use Cauchy's Integral form. By the method of partial fractions (Do yourself)

$$\frac{1}{z^3+z} = \frac{1}{z(z+i)(z-i)} = \frac{1}{z} - \frac{1}{2(z+i)} - \frac{1}{2(z-i)}$$

$$\therefore \int_\gamma \frac{\cos z}{z^3+z}\,dz = \int_\gamma \frac{\cos z}{z}\,dz - \frac{1}{2}\int \frac{\cos z}{z+i}\,dz - \frac{1}{2}\int \frac{\cos z}{z-i}\,dz$$

$\because z=0, i, -i$ lies inside γ, by Cauchy's Integral formula.

$$\int_\gamma \frac{\cos z}{z^3+z}\,dz = 2\pi i\left(\cos(0) - \frac{1}{2}\cos(-i) - \frac{1}{2}\cos(i)\right)$$
$$= 2\pi i\,(1-\cos i) \text{ Ans.} \qquad \{\because \cos(-i) = \cos i\}$$

Ex-3. $\int_\gamma \frac{z^2}{z^2+1}\,dz \quad \gamma : |z|<1$ $\{\because z=\pm i$ does not lie in the region

as $f(z)$ is analytic on simply connected region $|z|<1$, so by Cauchy's theorem

$$\int_\gamma \frac{z^2}{z^2+1}\,dz = 0$$

Ex-4 $\int_\gamma \frac{1+e^z}{z}\,dz \quad , \gamma : |z|<1$

$z=0$ lies inside γ so using Cauchy's integral formula

$$\int_\gamma \frac{1+e^z}{z}\,dz = 2\pi i \times f(0) = 2\pi i \times \{1+e^0\} = 4\pi i$$

Ex-5. Evaluate $\int_\gamma \frac{e^{z^2}}{(z-1)^3}\,dz \quad , \gamma : |z|<2$

as denominator is not in linear factors so we can use Cauchy's integral formula for derivatives i.e.

$$f^n(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(s)\,ds}{(s-z)^{n+1}}$$

comparing this formula to this problem

$n=2$, $f(s)=e^{s^2}$ and $z=1$ (lies inside γ)

$$\int_\gamma \frac{e^{z^2}}{(z-1)^3}\,dz = \frac{2\pi i\, f''(1)}{2!} = \frac{\cancel{2}\pi i}{\cancel{2}} \times 6e = 6\pi i e. \text{ Ans}$$

$\{ f'(s) = e^{s^2}.2s$

$f''(s) = 2[s.e^{s^2}.2s + e^{s^2}]$

$= 2e^{s^2}[2s^2+1]$

$f''(1) = 2e[2+1]$

$= 6e$

Ex-6 Let $a, b \in \mathbb{C}$ with $|a| \neq 1$, $|b| \neq 1$

evaluate $\int_\gamma \frac{1}{(z-a)(z-b)}\,dz \quad , \gamma : |z|<1$

Soln Case-i $a, b \notin$ Interior(γ), then

$$\int_\gamma \frac{1}{(z-a)(z-b)}\,dz = 0 \quad \{\text{by Cauchy's theorem}\}$$

Case-ii If $a \notin Int(\gamma)$, $b \in Int(\gamma)$ then we can see integration as $\int_\gamma \frac{1/(z-a)}{(z-b)} dz$, $f(z) = \frac{1}{z-a}$ is analytic within and on γ

then by Cauchy's ~~Integral~~ formula,

$$\int_\gamma \frac{1/(z-a)}{(z-b)} dz = 2\pi i f(b) = \frac{2\pi i}{b-a} \qquad \left\{ f(b) = \frac{1}{b-a} \right\}$$

Case-iii $b \notin Int(\gamma)$, $a \in Int(\gamma)$, similarly as in case (ii), replacing a by b

$$\int_\gamma \frac{1/(z-b)}{(z-a)} dz = 2\pi i f(a) = \frac{2\pi i}{a-b}$$

Case-iv. $a, b \in Int(\gamma)$, then by partial fraction

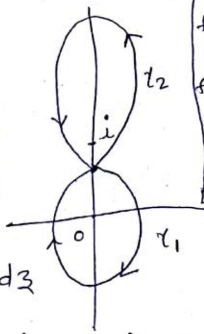
$$\frac{1}{(z-a)(z-b)} = \frac{1}{(a-b)}\left[\frac{1}{(z-a)} - \frac{1}{(z-b)}\right]$$

applying Cauchy's Integral formula $\left\{ f(z) = 1 \text{ here} \right\}$

$$\int_\gamma \frac{dz}{(z-a)(z-b)} = \frac{1}{a-b}[2\pi i - 2\pi i] = 0$$

Cauchy's Integral formula for derivatives

Ex.7 Using Cauchy's Integral formula to evaluate $\int_\gamma \frac{z^3+3}{z(z-i)^2} dz$ where γ is a contour given in fig.



for I_1

$f(z) = \frac{z^3+3}{z}$

$f'(z) = 2z - \frac{3}{z^2}$

$f'(i) = 2i + 3$

Sol^n γ is not simple closed contour, so we can think it as union of two simple closed contours γ_1 and γ_2 but γ_1 is in (-)ve direction, Hence

$$\int_\gamma \frac{z^3+3}{z(z-i)^2} dz = \int_{\gamma_2} \frac{z^3+3}{z(z-i)^2} dz \underset{\text{due to direction}}{(-)} \int_{\gamma_1} \frac{z^3+3}{z(z-i)^2} dz$$

$$= \int_{\gamma_2} \frac{\{(z^3+3)/z\}}{(z-i)^2} dz - \int_{\gamma_1} \frac{(z^3+3)/(z-i)^2}{z} dz$$

$$= \frac{2\pi i}{1} f'(i) - 2\pi i f(0) = -4\pi + 6\pi i - 2\pi i(-3)$$

$$= -4\pi + 12\pi i \quad \text{Ans}$$

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Theorem. Morera's theorem.

Statement. If a function $f(z)$ is continuous throughout a domain D and $\int_\gamma f(z)\,dz = 0$ for every closed contour γ in D, then $f(z)$ is analytic throughout D.

Proof. Given that $f(z)$ is contin. in D.

and $\int_\gamma f(z)\,dz = 0 \Rightarrow$ Integral is path independent

$\Rightarrow f(z)$ has antiderivative $F(z)$ in D

i.e. $F'(z) = f(z) \quad \forall z \in D$

$\Rightarrow F(z)$ is analytic in D {as derivative exist at ~~each~~ nbd of the each $z \in D$}

Again we know the result that derivatives of each order of an analytic fns are also analytic

$\Rightarrow$ If $F(z)$ is analytic then $F'(z) = f(z)$ is also analytic in D.

Lecture - 4

* **Note:** derivatives of all orders of an analytic fns exist and each order derivative is analytic also.

What about real valued functn ??
(fns of real variables)

Ex:-
Let $f(x) = (x+1)^{5/3}$ for $x \in \mathbb{R}$, then

$f'(x) = \frac{5}{3}(x+1)^{2/3}$ $\quad f'(-1) = 0$ — exist

$f''(x) = \frac{10}{9}(x+1)^{-1/3}$ $\quad$ for $x \neq -1$ — does not exist

for the complex valued same fn.

let $f(z) = (z+1)^{5/3}$, $z = -1$ is a branch pt. and analytic branch of $f(z)$ exists in $\mathbb{C} - \{x + i0 : x \leq -1\}$

Consequences of the Cauchy Integral formula

① **Cauchy's Inequality**

Let $f(z)$ be analytic in a domain D and $\gamma = \{z : |z-a| = R\}$ contained in D. then

$$|f^{(n)}(a)| \le \frac{n!\, M_R}{R^n} \qquad n = 0, 1, 2, \ldots$$

where $M_R = \max_{z \in \gamma} |f(z)|$

Proof. As we know tat derivatives of all orders of an analytic fns exist in interior of γ.

and

$$f^n(a) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}}\, dz$$

taking absolute value each side

$$|f^n(a)| = \frac{n!}{2\pi} \left| \int_\gamma \frac{f(z)}{(z-a)^{n+1}}\, dz \right| \qquad \{\because |i| = 1\}$$

$$\le \frac{n!}{2\pi} \left| \int_\gamma \frac{|f(z)|}{|(z-a)|^{n+1}} |dz| \right| \qquad \left\{\because \int_\gamma |dz| = 2\pi R\right\}$$

$$\le \frac{n!}{2\pi} \times \frac{M_R}{R^{n+1}} \cdot 2\pi R = \frac{n!\, M_R}{R^n}$$

proved.

Remark 1: The number M_R depends on the circle $|z-a| = R$

But for $n = 0$, we have

$$|f(a)| \le M_R$$

$\Rightarrow$ upper bound M_R of $|f(z)|$ on any circle about a cannot be smaller than $|f(a)|$

Remark 2: Is cauchy's Inequality holds for real valued fns (fns of real variables)

Solⁿ. Ex: let $u_n(x) = \sin nx$ then for each $n \in \mathbb{Z}$

$|u_n(x)| \le 1$ on $\mathbb{R}$

now $u_n'(x) = n \cos nx$ exist $\forall x \in \mathbb{R}$ and $u_n'(0) = n$ (unbounded) $\mathbb{R}$, on

Liouville's Theorem

Th// A bounded entire function is constant.

Proof function f is entire $\Rightarrow$ f is analytic $\forall$ points in $\mathbb{C}$.

Let z be any arbitrary point.

By Cauchy's inequality

$$|f^n(z)| \le \frac{n!\,M_R}{R^n} \qquad \because f(z) \text{ is bounded} \Rightarrow M_R \le M \text{ for any } R \text{ s.t. } \{~~|z|~~ = |s-z| = R\}$$

for $n = 1$

$$|f'(z)| \le \frac{M}{R} \quad \text{holds } \forall R > 0$$

let $R \to \infty$ then $|f'(z)| = 0 \Rightarrow f'(z) = 0 \ \forall z$

Hence $f(z)$ is constant.

i.e The Range of a ~~not~~ bounded and entire fns is a singleton set.

Corr// $\boxed{\text{A non constant entire function is unbounded.}}$

Ex: $\sin z$ and $\cos z$ are entire, non constant and is unbounded.

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \text{and} \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

for $z = iy$, $y \in \mathbb{R}$

$$\cos iy = \frac{e^{-y} + e^{y}}{2} \quad \text{and} \quad |\sin(iy)| = \frac{|e^{-y} - e^{y}|}{2}$$

$\underbrace{\cos z \ \& \ \sin z}_{\text{increases indefinitely}}$ when $z \to \infty$ along the imaginary axis.

Corr-2 Every analytic function in the extended complex plane is necessarily constant.

f is analytic at $z = \infty \Rightarrow \lim_{|z|\to\infty} f(z)$ is finite.

let this limit be L.

i.e. $\forall\, \varepsilon > 0\ \exists$ an $R > 0$ s.t.

$|f(z)| - |L| \leq |f(z) - L| < \varepsilon$ whenever $|z| > R$

$\Rightarrow f$ is bounded on the entire plane.

Hence by Liouville's theorem,

f is constant.

i.e. the only function which is analytic on the Riemann sphere is the constant fns.

Corr-3 If f is entire and $\exists\, M > 0$ with $|f(z)| > M$ for all $z \in \mathbb{C}$, then f is constant.

f is analytic and $f(z) \neq 0$ in $\mathbb{C}$

so that $\frac{1}{f(z)}$ is analytic on $\mathbb{C}$

and $\left|\frac{1}{f(z)}\right| < \frac{1}{M}\ \forall\, z \in \mathbb{C}$

now by Liouville's th.

Every bounded entire fns must be const.

$\Rightarrow \frac{1}{f(z)}$ is const. $\Rightarrow f(z)$ is constant.

Note: Boundedness condition in Liouville's theorem can be replaced by

(i) $\text{Re} f(z)$ or $\text{Im} f(z)$ is bounded on $\mathbb{C}$

(ii) $\text{Re} f(z)$ or $\text{Im} f(z)$ lies in a half plane.

for ex. If f is entire and $\text{Re} f(z) \leq M$ for some fixed $M \in \mathbb{R}$, then f is bounded

$f(z)$ is entire $\Rightarrow \phi(z) = e^{f(z)}$ is entire

$$\Rightarrow |\phi(z)| = |e^{f(z)}| = |e^{u+iv}| = e^{\text{Re} f(z)} \cdot \underbrace{|e^{iv}|}_{\leq 1}$$

$$\leq e^{M} \quad \forall z \in \mathbb{C}$$

$\Rightarrow \phi(z)$ is const. {bounded entire fns}

$\Rightarrow \phi'(z) = e^{f(z)} \cdot f'(z) = 0 \quad \forall z \in \mathbb{C}$

$\Rightarrow f'(z) = 0 \quad \{\because e^{f(z)} \neq 0\}$

$\Rightarrow f(z)$ is const.

Ex. Let $f(z)$ and $g(z)$ are entire functions; $g(z) \neq 0$ and $|f(z)| \leq |g(z)| \ \forall z$. Show that there is a const. c s.t. $f(z) = c\,g(z)$

Soln Let us define the fns $h(z) = \dfrac{f(z)}{g(z)}$ (well defined)

$h(z)$ is entire fns and also $|h(z)| \leq 1$

$\Rightarrow h(z)$ is const fns

$\Rightarrow h(z) = C \Rightarrow f(z) = c\,g(z)$

Fundamental Theorem of algebra

Let $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$, $a_n \neq 0$
then there exist a point $z_0 \in \mathbb{C} \ni P(z_0) = 0$

Proof: Let $P(z) = a_0 + a_1 z + \cdots + a_n z^n$, $a_n \neq 0$, $n \geq 1$
Assume that $P(z)$ has no zeros in $\mathbb{C}$.
i.e. $P(z) \neq 0 \ \forall z$.
consider the fns $Q(z) = \frac{1}{P(z)}$ well defined

$\because P(z)$ is non const. entire fns $\Rightarrow P(z)$ is unbounded

$\Rightarrow \lim\limits_{z \to \infty} P(z) \to \infty$ or $\underbrace{\lim\limits_{z \to \infty} Q(z) \to 0}_{\text{using def}^n \text{ of limit}}$

there exist $R > 0$ s.t.
$|Q(z) - 0| < \varepsilon$ whenever $|z| > R$
y∞

and $Q(z)$ is bounded in $|z| \leq R$ {as $Q(z)$ is contin fns}
$\Rightarrow Q(z)$ is bounded $\forall z \in \mathbb{C}$
Being bounded entire fns $Q(z)$ must be const.
[Liouville th.]

$\Rightarrow P(z)$ is const. [contradict]
hence $\exists z_0 \in \mathbb{C} \ni P(z_0) = 0$

Corr. A polynomial of degree n has exactly n zeroes (roots).

Proof: The fundamental theorem of algebra states that a polyn. of degree n has atleast one zero.
consider the n^{th} degree polyn. of the form
$$P_n(z) = b_0 + b_1 z + \cdots z^n \quad n \geq 1$$
Let $\exists$ atleast one $z_1 \ni P_n(z_1) = 0$

By division algorithm, there exist a polyn. of degree $(n-1)$ such that

$$P_n(z) = (z - z_1)\, P_{n-1}(z)$$

Thus if $n > 1$, we can again apply fundamental theorem of algebra to say that there exist a $z_2 \in \mathbb{C}$ s.t. $P_{n-1}(z_2) = 0$, again we can write

$$P_n(z) = (z - z_1)(z - z_2)\, P_{n-2}(z)$$

likewise we can express $P(z)$ uniquely as

$$P_n(z) = (z - z_1)(z - z_2) \cdots (z - z_n)$$

$\Rightarrow \exists$ exactly n zeroes (may not be distinct)

Note: $\rightarrow$ fundamental th. of algebra holds for real no ??.
{Hint: $\rightarrow x^2+1$}

Definition

Convex hull :– The convex hull of a set D in $\mathbb{C}$, is the intersection of all convex sets containing D.]

Proposition (Gauss) Let $p(z)$ is a polynomial of degree $n \geq 1$. Then every zero of $p'(z)$ lies in the convex hull of the set of zeros of $p(z)$
[Statement only]

Theorem [Luca] If all zeros of a polynomial lie in a half plane, then the zeros of the derivative also lie in the same half plane. [As half plane is a convex set so smallest convex set containing that half plane is half plane itself
So the th. [using above propositn]

THANK YOU !