Cyclic Subspace

Let T be a linear operator on a vector space V over field F. Let $0 \neq v \in V$. The subspace W spanned by the set $\{v, T(v), T^2(v), \ldots\}$ is called the T-cyclic subspace of V generated by v. That is, The set of all vectors of the form $f(T)v$ where $f(t)$ is a polynomial of F is the T-cyclic subspace of V generated by v. It is denoted by $Z(v,T) = W$. v is called cyclic vector for T.

To show W is T-invariant subspace of V.

Let $w \in W$, Then $w = \alpha_i T^i(v) + \alpha_j T^j(v) + \cdots + \alpha_k T^k(v)$.

$$\therefore T(w) = \alpha_i T^{i+1}(v) + \alpha_j T^{j+1}(v) + \cdots + \alpha_k T^{k+1}(v).$$

$$\Rightarrow T(w) \in W$$

So W is a T-invariant subspace of V.

If W' is a T-invariant subspace of V containing $v \in V$. Then $T(W') \subseteq W'$ and $v \in W'$

$$\Rightarrow T(v) \in W'$$

$$\Rightarrow T^r(v) \in W', \text{ for all integer } r > 0.$$

Let $w \in W$. Then $w = \alpha_i T^i(v) + \alpha_j T^j(v) + \cdots \alpha_k T^k(v)$

$$\Rightarrow w \in W' \quad [\because T^r(v) \in W' \text{ for all integer } r > 0.$$

$$\therefore W \subseteq W'$$

Hence W is the smallest subspace generated by v, which is T-invariant.

Also $W = \cap W'$

$\therefore$ $W = Z(v,t)$ is the intersection of all T-invariant subspaces of V containing v.

\# Let k be the least +ve integer such that $T^k(v)$ is a linear combination of ~~those~~ vectors $v, T(v), \ldots, T^{k-1}(v)$ i.e.

$$T^k(v) = -a_{k-1}T^{k-1}(v) - \cdots - a_1 T(v) - a_0 v$$

Then $m_v(t) = t^k + a_{k-1}t^{k-1} + \cdots + a_1 t + a_0$, is the unique monic polynomial of lowest degree s.t. $m_v(T)(v) = 0$ i.e. $m_v(t)$ is the T-annihilator of v and $Z(v,T)$.

Examples:-

(1) If $v=0$, $Z(0,T)=\{0\}$

(2) $Z(v,I)=$ Span$\{v\}$, $v\neq 0$.

(3) $Z(v,T)=$ Span$\{v\}$ iff v is an eigenvector for T.

Solⁿ:- Let $Z(v,T)=$ span$\{v\}$

Then $f(T)(v)=\alpha v$, $\alpha\in F$, for any polynomial $f(t)$.

In particular, $f(t)=t$, we have

$$T(v)=\alpha v$$

$\Rightarrow$ v is an eigen vector of T.

Conversely, if $Tv=\lambda v$ and if $f\in F[t]$

then $f(T)(v)=f(\lambda)v=\alpha v\in$ Span$\{v\}$

$\therefore$ $Z(v,T)=$ Span$\{v\}$.

(4) $T:\mathbb{R}^2\to\mathbb{R}^2$ is defined by $T(x)=(x_2,0)$, $x\in\mathbb{R}^2$ ie $x=(x_1,x_2)$

$e_1=(1,0)$, $e_2=(0,1)$

$Z(e_1,T)=\{f(T)e_1 : f\in F[t]\}$.

By definition, $T^2(x)=(0,0)=T^2=0$.

$$f(t)=\alpha_0+\alpha_1 t+\cdots+\alpha_r t^r$$

$$\Rightarrow f(T)=\alpha_0 I+\alpha_1 T$$

$$\Rightarrow f(T)(e_1)=\alpha_0 e_1+\alpha_1 T(e_1)$$

$$=\alpha_0 e_1+\alpha_1 e_2\in \text{Span}\{e_1,e_2\}$$

So $Z(e_1,T)=\mathbb{R}^2$

(5) Let T be a linear operator on a two-dimensional vector space V. Prove that either V is a T-cyclic subspace of itself or $T=cI$ for some scalar c.

Solⁿ:- We have done earlier that if every non-zero vector in V is an eigenvector of T, then $T=cI$, for some scalar c.

ie $T(v)=cv$ for all $v\in V$.

$\Rightarrow$ $\boxed{T=cI}$

Let $0 \neq v \in V$ be not an eigenvector of T.

Suppose $W = \langle v \rangle$, $\dim W = 1$

If $T(v) = v'$, then $v' \notin W$ since v is not an eigenvector of T.

Let $W' = \langle v' \rangle$. Then clearly $W \cap W' = \{0\}$

$\dim W' = 1$ { As, $w \in W \cap W' \Rightarrow w \in W$ & $w \in W'$

$\Rightarrow w = \alpha v$ $w = \beta v'$.

$\Rightarrow \alpha v = \beta v' \Rightarrow v' = \frac{\alpha}{\beta} v$,

a contradiction }

$\therefore V = W \oplus W'$

Now, Let $u \in V$. Then $u = \alpha v + \beta v'$

$\Rightarrow (\beta T + \alpha I)(v) = \beta T(v) + \alpha v$

$= \beta v' + \alpha v = u$

$\Rightarrow u = \beta T(v) + cv \in Z(v, T)$

$\therefore V \subseteq Z(v, T) \subseteq V$

$\Rightarrow V = Z(v, T)$

Thus V is a T-cyclic subspaces of itself.

<u>Theorem:</u> If $\dim Z(v, T) = n$, then show that $Z(v, T) = \langle v, T(v), \cdots, T^{n-1}(v) \rangle$.

<u>Proof:</u> Let $\dim Z(v, T) = n$, then any $n+1$ vectors of $Z(v, T)$ are linearly dependent.

Now $v, T(v), T^2(v), \cdots, T^n(v) \in Z(v, T) = W$ (say)

So there exists scalars $\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_n$ such that

$\alpha_0 v + \alpha_1 T(v) + \cdots + \alpha_n T^n(v) = 0$, for at least one $\alpha_i \neq 0$

If $\alpha_n \neq 0$, then $T^n(v)$ is a linear combination of vectors $v, T(v), \cdots, T^{n-1}(v)$.

So $w \in W \Rightarrow w$ is a linear combination of vectors $v, T(v), \cdots, T^{n-1}(v)$.

If $\alpha_n = 0$, then $\alpha_0 v + \alpha_1 T(v) + \cdots + \alpha_{n-1} T^{n-1}(v) = 0$, for some $\alpha_i \neq 0$

Thus any element $w \in W$ is a linear combination of vectors $v, T(v), \cdots, T^{n-1}(v)$.

So $W = \langle v, T(v), \cdots T^{n-1}(v) \rangle$.

Problem: Let T be a linear operator on $\mathbb{R}^3$ which is represented in the standard ordered basis by the matrix
$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. Prove that T has no cyclic vector.
What is $Z(v,T)$, $v=(1,-1,3)$?

Solution: Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be an operator and $(a,b,c)\in\mathbb{R}^3$ be a cyclic vector. Then $\mathbb{R}^3 = Z((a,b,c),T)$.
We have from matrix representation,

$$T(1,0,0) = (2,0,0) = 2(1,0,0)$$
$$T(0,1,0) = (0,2,0) = 2(0,1,0)$$
$$T(0,0,1) = (0,0,-1) = -1(0,0,1).$$
$$\therefore T(a,b,c) = aT(1,0,0) + bT(0,1,0) + cT(0,0,1)$$
$$= 2a(1,0,0) + 2b(0,1,0) + (-1)c(0,0,1)$$
$$= (2a, 2b, -c).$$

Similarly, $T^2(a,b,c) = (2^2 a, 2^2 b, c)$

Now, $(0,1,0) = f(T)(a,b,c)$
$$= (\alpha_2 T^2 + \alpha_1 T + \alpha_0 I)(a,b,c)$$
$$= (\alpha_2 2^2 a + \alpha_1 2a + \alpha_0 a,\ \alpha_2 2^2 b + \alpha_1 2b + \alpha_0 b,\ \alpha_2 c - \alpha_1 c + \alpha_0 c).$$

$\Rightarrow a(2^2\alpha_2 + 2\alpha_1 + \alpha_0) = 0 \Rightarrow a = 0.$
and $b(2^2\alpha_2 + 2\alpha_1 + \alpha_0) = 1$

$\therefore (1,0,0) = (\beta_2 T^2 + \beta_1 T + \beta_0 I)(0,b,c).$
$$= \beta_2 T^2(0,b,c) + \beta_1 T(0,b,c) + \beta_0 I(0,b,c)$$
$$(1,0,0) = (0, \beta_2 2^2 b + 2b\beta_1 + \beta_0 b, \beta_2 c - \beta_1 c + \beta_0)$$
$\Rightarrow 1 = 0$, a contradiction. So T has no cyclic vector.

Now, $Z((1,-1,3),T) = W.$
$$\therefore W = \{g(T)(1,-1,3) : g\in\mathbb{R}[x],\ \deg g(x) < 3\}$$
$$= \{(\alpha_2 T^2 + \alpha_1 T + \alpha_0 I)(1,-1,3) : \alpha_i \in \mathbb{R}\}$$
$$= \{4\alpha_2 + 2\alpha_1 + \alpha_0, -4\alpha_2 - 2\alpha_1 - \alpha_0, 3\alpha_2 - 3\alpha_1 + 3\alpha_0 ; \alpha_i \in \mathbb{R}\}$$
$$= \langle (4,-4,3), (2,-2,-3), (1,-1,3) \rangle.$$

But $(4,-4,3) = 1(2,-2,-3) + 2(1,-1,3)$.
$$\therefore W = \langle (2,-2,-3), (1,-1,3) \rangle.$$
$$\therefore \dim W = 2.$$

Problem: Prove that if T^2 has a cyclic vector, then T also has a cyclic vector. Is the converse true?

Solution: Let $V = Z(v, T^2)$. Then $u \in V$ implies that

$$u = \alpha_0 v + \alpha_2 T^2(v) + \cdots + \alpha_{2m} T^{2m}(v) \in Z(v, T)$$

$$\therefore \; V \subseteq Z(v, T) \subseteq V$$

$$\Rightarrow V = Z(v, T).$$

Thus v is also a cyclic vector of T.

The converse may not be true. We prove this by an example.

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be an linear operator defined by

$$T(1,0) = (0,1), \quad T(0,1) = (0,0).$$

Then $T^2(1,0) = (0,0) = T^2(0,1)$

So $T^2 = 0$.

$\therefore \; Z(v, T^2) = \langle v \rangle \neq \mathbb{R}^2$, for any $v \in V$.

$= Z(v, I) \Rightarrow T^2 = I$, a contradiction

Hence, T^2 has no cyclic vector.

Now, let $(a, b) \in \mathbb{R}^2$

$$\therefore \; (a, b) = a(1,0) + b(0,1)$$

$$= (a + bT)(1,0)$$

Thus $\mathbb{R}^2 = Z((1,0), T)$.

So T has a cyclic vector $(1,0)$.

Definition: The T-annihilator of $v \in V$ is the subspace

$$S_T(v, \{0\}) = \{ g \in F[t] : g(T)v = 0 \}.$$

We know that $m_v(t)$ is the T-annihilator of v,

Theorem: If V is T-cyclic subspace of dimension n, then the characteristic polynomial of T is same as the minimal polynomial for T.

Proof: Let $V = Z(v, T)$, $\dim V = n$.

Then $\beta = \{v, T(v), \ldots, T^{n-1}(v)\}$ spans V

Since $\dim V = n$, β is a basis of V.

Let $g(x) = a_0 + a_1 x + \cdots + a_k x^k$ be such that

$$\deg g(x) = k < n$$

Then $a_k \neq 0$.

$$\therefore \quad g(T)v = (a_0 I + a_1 T + \cdots + a_k T^k)(v)$$
$$= a_0 v + a_1 T(v) + \cdots + a_k T^k(v)$$
$$g(T)v \neq 0 \quad (\text{As } g(T)v = 0 \Rightarrow a_k = 0, \text{ a contradiction})$$
$$\Rightarrow g(T) \neq 0.$$

$\therefore$ T does not satisfy any polynomial of degree less than n.

But T satisfies its characteristic polynomial whose degree must be n.

$\therefore$ Minimal polynomial for T is same as the characteristic polynomial for T.

Theorem: Cayley-Hamilton Theorem

Let T be a linear operator on a finite dimensional vector space V. Let $f(x)$ be the characteristic polynomial for T. Then $f(T) = 0$.

Proof: Let W be the T-cyclic subspace generated by $v \in V$, $v \neq 0$ and $\dim W = k$.

Let $\beta = \{v, T(v), \ldots, T^{k-1}(v)\}$ be a basis of W.

Then there exist scalars $\alpha_0, \alpha_1, \ldots \alpha_k$ such that

$$\alpha_0 v + \alpha_1 T(v) + \cdots + \alpha_k T^{k}(v) = 0, \quad \text{for some } \alpha_i \neq 0.$$

If $\alpha_k = 0$ then $\alpha_i = 0 \ \forall i$. So $\alpha_k \neq 0$, α_k^{-1} exists.

$$\therefore \quad (\alpha_k^{-1}\alpha_0)v + (\alpha_k^{-1}\alpha_1)T(v) + \cdots + \alpha_k^{-1}\alpha_{k-1} T^{k-1}(v) + T^k(v) = 0$$
$$\Rightarrow a_0 v + a_1 T(v) + \cdots + a_{k-1} T^{k-1}(v) + T^k(v) = 0$$

Let T_W be the restriction of T on W. So

$$T_W = T \quad \text{for all } w \in W.$$

Since $v, T(v), \cdots T^{k-1}(v) \in W$. So

$$T_W(v) = T(v) = 0.v + 1.T(v) + \cdots + 0.T^{k-1}(v)$$

$$T_W(T(v)) = T(T(v)) = 0.v + 0.T(v) + 1.T^2(v) + \cdots + 0.T^{k-1}(v)$$

$$\cdots\cdots$$

$$T_W(T^{k-1}(v)) = T(T^{k-1}(v)) = T^k(v) = -a_0 v - a_1 T(v) + \cdots + (-a_{k-1}) T^{k-1}(v).$$

The matrix of T_W w.r.to basis β is the Companion matrix given by

$$[T_W]_\beta = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{k-1} \end{bmatrix} = A.$$

We know that the characteristic polynomial of companion matrix i.e Ch. polynomial for T_W is

$$g(x) = a_0 + a_1 x + \cdots + a_{k-1} x^{k-1} + x^k.$$

Since the characteristic polynomial $g(x)$ of T_W divides the characteristic polynomial $f(x)$ of T.

So $\quad f(x) = g(x) h(x) = h(x) g(x)$

$\Rightarrow f(T)v = h(T) g(T) v = 0 \quad \forall v \neq 0$ in V.

$\Rightarrow f(T) = 0 \quad \{ \because f(T)(0) = 0 \}$

* The minimal polynomial of T_W is same as the T-annihilator of v.

<u>Theorem</u>: Let T be a linear operator on an n-dimensional vector space V. Suppose that T is diagonalisable.

(i) If T has a cyclic vector, show that T has n distinct eigenvalues.

(ii) If T has n distinct eigenvalues and $\{v_1, v_2, \cdots, v_n\}$ is a basis of eigenvectors of T. Show that $v = v_1 + v_2 + \cdots + v_n$ is a cyclic vector of T.

<u>Proof</u>: (i) Let $T: V \to V$ be linear operator and $\dim V = n$.

Since T has a cyclic vector. So $V = Z(v, T)$, for some $v \in V$.

$\therefore$ Characteristic polynomial of T = minimal polynomial of T

Given T is diagonalisable, so min. polynomial is the product of distinct linear factors.

Since degree of ch. polynomial $= n$.
So degree of Min. polynomial $= n$.
$\Rightarrow$ T has n distinct eigenvalues.

(ii) Let T has n distinct eigenvalues, say $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then there exist eigenvectors $v_1, v_2, \ldots, v_n$ s.t.

$$T(v_1) = \lambda_1 v_1,\ T(v_2) = \lambda_2 v_2, \ldots, T(v_n) = \lambda_n v_n.$$

Let $v = v_1 + v_2 + \cdots + v_n$.

Let $S = \{v, T(v), \ldots, T^{n-1}(v)\}$

$$S = \{v_1 + \cdots + v_n,\ \lambda_1 v_1 + \cdots + \lambda_n v_n, \ldots, \lambda_1^{n-1} v_1 + \cdots + \lambda_n^{n-1} v_n\}.$$

Since $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct. S is a linearly independent set.

$\therefore$ S forms a basis of V.

Now, let $u \in V$, Then

$$u = \alpha_0 v + \alpha_1 T(v) + \cdots + \alpha_{n-1} T^{n-1}(v)$$
$$= (\alpha_0 + \alpha_1 T + \cdots + \alpha_{n-1} T^{n-1})(v)$$
$$u = g(T)v, \text{ where } g(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_{n-1} x^{n-1},$$

$\therefore$ v is a cyclic vector of T.

<u>Problem</u>: Let T be a linear operator on a finite dimensional vector space V. Show that T has a cyclic vector if and only if there exists an ordered basis β for V such that $[T]_\beta$ is the companion matrix of the minimal polynomial for T.