An Introduction to Hilbert Spaces

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Hilbert Spaces

Definition: Let X be a linear space over the field F. An inner product on X is a function $\langle \cdot , \cdot \rangle : X \times X \to F$ which satisfies the following conditions:

(i) $\langle x, x \rangle \geq 0 \quad , \quad \forall x \in X$

(ii) $\langle x, x \rangle = 0 \iff x = 0$

(iii) $\langle x, y \rangle = \overline{\langle y, x \rangle} \quad \forall x, y \in X$

(iv) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle \quad \forall x, y \in X$ and $\alpha \in F$

(v) $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle \; , \; \forall x, y, z \in X$

The scalar $\langle x, y \rangle$ is called the inner product of x and y, and X is called inner product space.

- The inner product space is also called pre-Hilbert space

Example: For vectors $x = (x_1, x_2, \ldots x_n)$ and $y = (y_1, y_2, \ldots y_n)$ in $\mathbb{R}^n$, define

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$$

The space $(\mathbb{R}^n, \langle \cdot , \cdot \rangle)$ is a real inner product space.

Example: For vectors $x = (x_1, x_2, \ldots x_n)$ and $y = (y_1, y_2, \ldots y_n)$ in $\mathbb{C}^n$ define

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y_i}$$

The space $(\mathbb{C}^n, \langle \cdot , \cdot \rangle)$ is a complex inner product space.

Example: Consider the linear space l^2. For vectors $x = \{x_i\}$ and $y = \{y_i\}$ in l^2, define

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \overline{y_i}$$

The space $(l^2, \langle \cdot, \cdot \rangle)$ is an inner product space.

Example: Consider the linear space $C[a,b]$, the set of all continuous real valued (or complex valued) function on $[a,b]$. For vectors $x, y \in C[a,b]$, define

$$\langle x, y \rangle = \int_a^b x(t)\, \overline{y(t)}\, dt$$

The space $(C[a,b], \langle \cdot, \cdot \rangle)$ is an inner product space

Example Consider the linear space l_2^n. For vectors $x = \{x_i\}$ and $y = \{y_i\}$ in l_2^n, define

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i \overline{y_i}$$

The space $(l_2^n, \langle\cdot,\cdot\rangle)$ is an inner product space.

proof:

(i) $\langle x, x\rangle = \sum_{i=1}^{n} x_i \overline{x_i} = \sum_{i=1}^{n} |x_i|^2 \geq 0$.

(ii) $\langle x, x\rangle = 0 \Leftrightarrow \sum_{i=1}^{n} |x_i|^2 = 0 \Leftrightarrow x_i = 0 \ \forall i$

$\Leftrightarrow x = 0$.

(iii)
$$\overline{\langle x, y\rangle} = \overline{\sum_{i=1}^{n} x_i \overline{y_i}}$$
$$= \overline{(x_1\overline{y_1} + x_2\overline{y_2} + x_3\overline{y_3} + \dots + x_n\overline{y_n})}$$
$$= \overline{x_1\overline{y_1}} + \overline{x_2\overline{y_2}} + \overline{x_3\overline{y_3}} + \dots + \overline{x_n\overline{y_n}}$$
$$= \overline{x_1} y_1 + \overline{x_2} y_2 + \overline{x_3} y_3 + \dots + \overline{x_n} y_n$$
$$= y_1\overline{x_1} + y_2\overline{x_2} + y_3\overline{x_3} + \dots + y_n\overline{x_n}$$
$$= \sum_{i=1}^{n} y_i \overline{x_i}$$
$$= \langle y, x\rangle$$

(iv)
$$\langle \alpha x + \beta y, z\rangle = \sum_{i=1}^{n} (\alpha x_i + \beta y_i)\overline{z_i}$$
$$= \sum_{i=1}^{n} \alpha x_i \overline{z_i} + \sum_{i=1}^{n} \beta y_i \overline{z_i}$$
$$= \alpha \sum_{i=1}^{n} x_i \overline{z_i} + \beta \sum_{i=1}^{n} y_i \overline{z_i}$$
$$= \alpha \langle x, z\rangle + \beta \langle y, z\rangle$$

Thus l_2^n is an inner product space.

Theorem If X is a complex inner product space, then

(i) $\langle \alpha x - \beta y, z \rangle = \alpha \langle x, z \rangle - \beta \langle y, z \rangle$.

(ii) $\langle x, \beta y + \gamma z \rangle = \bar{\beta} \langle x, y \rangle + \bar{\gamma} \langle x, z \rangle$.

(iii) $\langle x, \beta y - \gamma z \rangle = \bar{\beta} \langle x, y \rangle - \bar{\gamma} \langle x, z \rangle$

(iv) $\langle x, 0 \rangle = 0$ and $\langle 0, x \rangle = 0$ for every $x \in X$.

Proof: (i)
$$\begin{aligned}\langle \alpha x - \beta y, z \rangle &= \langle \alpha x + (-\beta) y, z \rangle \\ &= \langle \alpha x, z \rangle + \langle (-\beta) y, z \rangle \\ &= \alpha \langle x, z \rangle + (-\beta) \langle y, z \rangle \\ &= \alpha \langle x, z \rangle - \beta \langle y, z \rangle\end{aligned}$$

(ii)
$$\begin{aligned}\langle x, \beta y + \gamma z \rangle &= \overline{\langle \beta y + \gamma z, x \rangle} \\ &= \overline{\langle \beta y, x \rangle + \langle \gamma z, x \rangle} \\ &= \overline{\beta \langle y, x \rangle + \gamma \langle z, x \rangle} \\ &= \bar{\beta}\, \overline{\langle y, x \rangle} + \bar{\gamma}\, \overline{\langle z, x \rangle} \\ &= \bar{\beta} \langle x, y \rangle + \bar{\gamma} \langle x, z \rangle\end{aligned}$$

Hence $\langle x, \beta y + \gamma z \rangle = \bar{\beta} \langle x, y \rangle + \bar{\gamma} \langle x, z \rangle$.

(iii)
$$\begin{aligned}\langle x, \beta y - \gamma z \rangle &= \langle x, \beta y + (-\gamma) z \rangle \\ &= \bar{\beta} \langle x, y \rangle + \overline{(-\gamma)} \langle x, z \rangle \\ &= \bar{\beta} \langle x, y \rangle - \bar{\gamma} \langle x, z \rangle.\end{aligned}$$

(iv) $\langle 0, x \rangle = \langle 0.\mathbf{0}, x \rangle$ where $\mathbf{0}$ is the zero element of X.
$$\begin{aligned}&= 0 \langle \mathbf{0}, x \rangle \\ &= 0\end{aligned}$$

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An inner product on X generates a norm on X given by $\|x\| := +\sqrt{\langle x, x\rangle}$,
and a metric on X given by

$$d(x,y) := \|x-y\| = +\sqrt{\langle x-y,\ x-y\rangle}.$$

<u>Theorem (Cauchy Schwarz - inequality)</u>

If x and y are any two vectors in an inner product space, then

$$|\langle x,y\rangle| \leq \|x\|\ \|y\| \quad \text{- - - (1)}$$

where the equality holds if and only if x and y are linearly dependent.

<u>Proof</u>: If $y=0$, then both sides of the inequality (1) are zero and so the equality holds.
For every scalar α we have

$$0 \leq \|x-\alpha y\|^2 = \langle x-\alpha y,\ x-\alpha y\rangle$$
$$= \langle x,x\rangle - \bar{\alpha}\langle x,y\rangle - \alpha\left[\langle y,x\rangle - \bar{\alpha}\langle y,x\rangle\right]$$

Let us assume that $y \neq 0$ and choose $\bar{\alpha} = \dfrac{\langle y,x\rangle}{\langle y,y\rangle}$,

then the expression inside the square brackets vanish and we have

$$0 \leq \langle x,x\rangle - \frac{\langle y,x\rangle}{\langle y,y\rangle}\langle y,y\rangle = \|x\|^2 - \frac{\overline{\langle x,y\rangle}\langle x,y\rangle}{\|y\|^2}$$

$$0 \leq \|x\|^2 - \frac{|\langle x,y\rangle|^2}{\|y\|^2}$$ and therefore

$$|\langle x,y\rangle|^2 \leq \|x\|^2\ \|y\|^2$$

or $|\langle x,y\rangle| \leq \|x\| \, \|y\|$

Now, suppose that $|\langle x,y\rangle| = \|x\| \, \|y\|$. If either of x or y is zero then x and y are linearly dependent. So let $\|x\| \neq 0$, $\|y\| \neq 0$, then

$\langle x,x\rangle \neq 0$ and $\langle y,y\rangle \neq 0$.

We have already noted that for $\alpha = \dfrac{\langle x,y\rangle}{\langle y,y\rangle}$.

$$\|x-\alpha y\|^2 = \|x\|^2 - \frac{|\langle x,y\rangle|^2}{\|y\|^2}$$

$$\|x-\alpha y\|^2 = \|x\|^2 - \frac{\|x\|^2 \, \|y\|^2}{\|y\|^2} = 0$$

and therefore $x = \alpha y$.

This prove that x and y are linearly dependent

Conversely, if x and y are linearly dependent, then one of them is a scalar multiple of the other, say $y = \alpha x$. Then

$$|\langle x,y\rangle| = |\langle x, \alpha x\rangle| = |\bar{\alpha}| \, \|x\|^2$$
$$= |\alpha| \, \|x\|^2$$
$$= \|x\| \, |\alpha| \, \|x\|$$
$$= \|x\| \, \|\alpha x\| = \|x\| \, \|y\|$$

and so when x and y are linearly dependent equality in (1) holds.

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Theorem If X is an inner product space, then $\|x\| = \sqrt{\langle x, x\rangle}$ is a norm on X.

Proof: (i) $\|x\| \geq 0$ since $\langle x, x\rangle \geq 0$

(ii) $\|x\| = 0 \Leftrightarrow x = 0$ since $\langle x, x\rangle = 0 \Leftrightarrow x = 0$.

(iii) $\|\alpha x\|^2 = \langle \alpha x, \alpha x\rangle$.

$$= \alpha \bar{\alpha} \langle x, x\rangle$$

$$\|\alpha x\|^2 = |\alpha|^2 \|x\|^2$$

$$\Rightarrow \|\alpha x\| = |\alpha| \|x\|.$$

(iv) $\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle$

$$= \|x\|^2 + \langle x, y\rangle + \overline{\langle x, y\rangle} + \|y\|^2$$

$$= \|x\|^2 + 2\,\mathrm{Re}\,\langle x, y\rangle + \|y\|^2$$

$$\leq \|x\|^2 + 2\,|\langle x, y\rangle| + \|y\|^2$$

$$\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2$$ by Cauchy-Schwarz inequality,

i.e. $\|x+y\|^2 \leq (\|x\| + \|y\|)^2$

and therefore

$$\|x+y\| \leq \|x\| + \|y\|$$

Definition: A complete inner product space is called a Hilbert space.

Observe that from the definition it is clear that every Hilbert space is a Banach space. Later we shall show that there exist Banach space which are not Hilbert space.

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Theorem The inner product in a Hilbert space is jointly continuous. That is if $x_n \to x$ and $y_n \to y$, then $\langle x_n, y_n\rangle \to \langle x, y\rangle$ as $n \to \infty$.

Proof:
$$|\langle x_n, y_n\rangle - \langle x, y\rangle| = |\langle x_n, y_n\rangle - \langle x_n, y\rangle + \langle x_n, y\rangle - \langle x, y\rangle|$$
$$= |\langle x_n, y_n - y\rangle + \langle x_n - x, y\rangle|$$
$$\leq |\langle x_n, y_n - y\rangle| + |\langle x_n - x, y\rangle|$$
$$\leq \|x_n\|\,\|y_n - y\| + \|x_n - x\|\,\|y\| \quad \text{(by Schwarz inequality)}$$

Since $x_n \to x$ and $y_n \to y$, $\|x_n - x\| \to 0$ and $\|y_n - y\| \to 0$.
Further since $\{x_n\}$ is a convergent sequence, it is bounded so that $\|x_n\| \leq M$ for all n.

Thus, we get $\langle x_n, y_n\rangle \to \langle x, y\rangle$ as $n \to \infty$.
Hence the inner product in a Hilbert space is jointly continuous.

Theorem: For any two elements x and y in an inner product space X, the following known as parallelogram law holds:

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$$

Proof: we have

$$\|x+y\|^2 = \langle x+y, x+y\rangle$$
$$= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle$$
$$\|x+y\|^2 = \|x\|^2 + \langle x, y\rangle + \langle y, x\rangle + \|y\|^2 \quad --(1)$$

and
$$\|x-y\|^2 = \langle x, x\rangle - \langle x, y\rangle - \langle y, x\rangle + \langle y, y\rangle$$
$$= \|x\|^2 - \langle x, y\rangle - \langle y, x\rangle + \|y\|^2 \quad -(2)$$

Adding (1) and (2) we get, the parallelogram law follows:

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$$

The following example shows that for any arbitrary norm on a linear space the parallelogram law may not hold.

Example: Let $X = C[0, 2\pi]$, the space of all real-valued continuous functions on $[0, 2\pi]$, We know that $C[0, 2\pi]$ is a Banach space with the norm

$$\|f\| = \sup_{0 \leq t \leq 2\pi} |f(t)|$$

We show that this norm does not satisfy the parallelogram law.

Take $f(t) = \max(\sin t, 0)$, $g(t) = \max(-\sin t, 0)$

i.e. $f(t) = \dfrac{\sin t + |\sin t|}{2}$ and $g(t) = \dfrac{-\sin t + |\sin t|}{2}$

Clearly, $\|f\| = 1$, $\|g\| = 1$

$\therefore$ $\|f+g\| = 1$, $\|f-g\| = 1$.

Thus $2(\|f\|^2 + \|g\|^2) = 4$ But $\|f+g\|^2 + \|f-g\|^2 = 2$.

and so
$$\|f+g\|^2 + \|f-g\|^2 \neq 2(\|f\|^2 + \|g\|^2).$$

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Thank You!