Basic Probability Theory

Mr. Anup Singh
Department of Mathematics
Mahatma Gandhi Central university
Motihari-845401, Bihar, India
E-mail: anup.singh254@gmail.com

Experiment: It is an operation/action which produce well define result

There are two types.

(1) Deterministic experiment

(2) Random/Probabilistic experiment

Deterministic experiment: It is an experiment, when repeated under identical conditions, produce the same outcome every time, then such an experiment is known as a deterministic experiment.

Example – Predicting the amount of money in a bank account if you know the initial deposit, and the interest rate then you can determine the amount in the account after one year.

Random experiment: If an experiment, when repeated under idetical conditions, do not produce the same outcome every time but the outcome in a trial is one of the several possible outcomes then such an experiment is known as a random experiment.

Example - Tossing a coin, throwing a die.

Sample Space: The set of all possible outcomes of a random experiment is called the sample space associated with it and it is generally denoted by S.

Example (a) If a coin is tossed then

$S = \{H, T\}$

(b) If two coins are tossed together then

$S = \{H, T\} \times \{H, T\}$

Event: A subset of the sample space associated with a random experiment is called an event. It is denoted by E or A, B, C, ...

Example - Throwing a die.

$S = \{1, 2, 3, 4, 5, 6\}$ ⇐ Sample

$A = \{2, 4, 6\}$ ⇐ Event

Types of event:

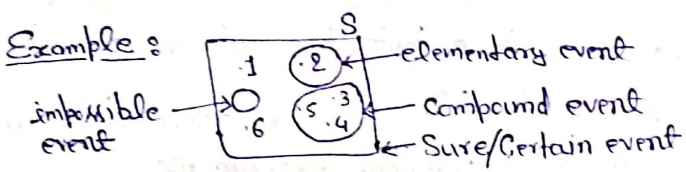
(1) **Elementary event** - An event which contain only one element is called elementary event.

(2) **Impossible event** - Empty subset of event is called impossible event. e.g. $\phi \subset S$

(3) **Compound event** - If an event contain more than one element is called compound event.

(4) **Certain/Sure event** - Sample space is also an event ($S \subseteq S$) is called certain/sure event.

Example:



Equally likely events -: A set of events are called equally likely events if the chances of ~~happing~~ happening of an event is not greater than or less than those of any other.

e.g. - In throwing an unbiased die, all the six faces are equally likely to come.

Mutually exclusive events: Two events A and B are said to be mutually exclusive events if the occurrence of any one of them excludes the occurrence of the other event i.e. they can not therefore occur simultaneously.

Example- (a) Tossing a coin the events head and tail are mutually exclusive.

(b) If a die is thrown then $S = \{1, 2, 3, 4, 5, 6\}$. A and B events are such that $A = \{1, 3\}$, $B = \{2, 4, 6\}$, $A \cap B = \phi$ are mutually exclusive.

Exhaustive events: A set of events is said to be exhaustive events if the performance of the experiment always result in the occurrence of at least one of them. Thus if a set of events $E_1, E_2, E_3, \ldots, E_n$ are subsets of a sample space S, they are said to be exhaustive events if

$$E_1 \cup E_2 \cup E_3 \cup \ldots \cup E_n = S.$$

Favourable elementary events: Let S be a sample space associated with a random experiment and A be an event associated with the experiment. Then elementary events belonging to A are known as favourable elementary elements to the event.

Definition of probability:

(1) Classical or Mathematical definition: If there are 'n' elementary events associated with a random experiment and 'm' of them are favourable to an event A, then the probability of occurence of A is denoted by P(A) and is defined as

$$P(A) = \frac{\text{Fourable no. of elementary events}}{\text{Total no. of elementary events}} = \frac{m}{n} = \frac{n(A)}{n(S)}$$

Note: $0 \le P(A) \le 1$.

Example: $S = \{1, 2, 3, 4, 5, 6\}$, $A = \{2, 4, 6\}$, $P(A) = \frac{3}{6} = \frac{1}{2}$.

Questions (1) What is the chance that a leap year selected at random will contain 53 sundays?

Solution- There are 366 days (leap year), therefore total no. of complete weeks are 52 and 2 days over. Possible combinations for these two over days are (i) {S, M} (ii) {M, T}, (iii) {T, W} (iv) {W, T}, (v) {T, F}, (vi) {F, S}, (vii) {S, S}.

Since ~~all of the~~ $n(S) = 7$, $n(E) = 2$

$\Rightarrow P(E) = \frac{2}{7}$.

Question (2) Each coefficient in equation $ax^2 + bx + c = 0$ is determined by throwing an ordinary die. Find the probability that the equation will have real roots.

Solution- The roots of the equation $ax^2 + bx + c = 0$ will be real if $b^2 \ge 4ac$.

Since each coefficients a, b and c determined by throwing a die i.e. it can take the values from 1 to 6.

Total number of possible outcomes $n(S) = 6 \times 6 \times 6 = 216$.

And no. of favourable cases can be enumerated as follow:

a.c	a	c	4ac	b (so that $b^2 \ge 4ac$)	No. of cases
1	1	1	4	{2, 3, 4, 5, 6}	$1 \times 5 = 5$
2	{1/2}	{2/1}	8	{3, 4, 5, 6}	$2 \times 4 = 8$
3	(i) 1 (ii) 3	3 1	12	{4, 5, 6}	$2 \times 3 = 6$
4	(i) 1 (ii) 2 (iii) 4	4 2 1	16	{4, 5, 6}	$3 \times 3 = 9$

ac	a	c	4ac	b (So that $b^2 \geq 4ac$)	No. of cases
5	(i) 1	5	20	{5, 6}	$2 \times 2 = 4$
	(ii) 5	1			
6	(i) 1	6	24	{5, 6}	$4 \times 2 = 8$
	(ii) 2	3			
	(iii) 3	2			
	(iv) 6	1			
8	(i) 2	4	32	{6}	$2 \times 1 = 2$
	(ii) 4	2			
9	(i) 3	3	36	{6}	$1 \times 1 = 1$
					$n(E) = 43$

$$P(E) = \frac{n(E)}{n(S)} = \frac{43}{216} = 0.199 \text{ Answer}$$

Question -(3) (a) Four cards are drawn at random from a pack of 52 cards. Find the probality that

(i) They are a king, a queen, a jack and an ace.
(ii) Two are kings and two are queens.
(iii) Two are black and two are red.
(iv) There are two cards of hearts and two cards of diamonds.

(b) In shuffling a pack of cards, four are accidentally dropped find the chance that the missing cards should be one from each suit.

Solution - Four cards can be drawn from a well-shuffled pack of 52 cards in $^{52}C_4$ ways.

(i) 1 king can be drawn out of the 4 kings in $^4C_1$ ways. Similarly 1 queen, 1 Jack and 1 ace can each be drawn in $^4C_1$ ways. Since any one of the ways of drawing a king can be associated with any one of the ways of drawing a queen, a jack and an ace, the favourable no. of cases are $^4C_1 \times {^4C_1} \times {^4C_1} \times {^4C_1}$

Hence required probability $= \frac{^4C_1 \times {^4C_1} \times {^4C_1} \times {^4C_1}}{^{52}C_4}$

(ii) Required probality $= \frac{^4C_2 \times {^4C_2}}{^{52}C_4}$.

(iii) Since there are 26 black cards (of spades and clubs) and 26 red cards (of diamonds and hearts) in a ~~ka~~ pack of cards, the required probability = $\frac{^{26}C_2 \times {}^{26}C_2}{^{52}C_4}$

(iv) Required probability = $\frac{^{13}C_2 \times {}^{13}C_2}{^{52}C_4}$

(b) There are $^{52}C_4$ possible ways in which four cards can slip which shuffling a pack of cards. The favourable no. of cases in which the four cards can be one from each suit is $^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1$.

$\therefore$ The required probability = $\frac{^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1}{^{52}C_4} = \frac{2197}{20825}$.

Question (4) n persons are ~~selected~~ seated on n chairs at a random table. Find the probability that two specified persons are sitting next to each other.

Solution – Since n persons can be seated in n chairs at a round table in $(n-1)!$ ways, the exhaustive no. of cases = $(n-1)!$

Assuming the two specified persons A and B who sit together as one, we get $(n-1)$ persons in all, who can be seated at a round table in $(n-2)!$ ways. A and B can interchange their positions in $2!$ ways, total no. of favourable cases of getting A and B together is $(n-2)! \times 2!$

$\therefore$ Required probability = $\frac{(n-2)! \times 2!}{(n-1)!} = \frac{2}{(n-1)}$.

Question (5) An urn contains 6 white, 14 red and 9 black balls. If 3 balls are drawn at random, find the probability that (i) two of the balls drawn are white, (ii) One is of each colour, (iii) none is red (iv) at least one is white.

Question (6) A six-figure no. is formed by the digits 0, 1, 2, 3, 4, 5 (without repetition). Find the probability that the number formed is divided by 4.

Question-(7) If the letters of the word REGULATIONS be arranged at random, what is the chance that there will be exactly 4 letters between R and E?

Question-(8) 50 books are placed at random in a shelf. Find the probability that a particular pair of books shall be (i) Never together (ii) Always together.

Question-(9) A car is parked among N cars in a row, not at either end. On his return the owner finds that exactly r of the N places are still occupied. What is the probability that both neighbouring places are empty?

Question-(10) What is the probability that at least two out of n people have the same birthday? Assume 365 days in a year and that all days are equally likely.

Reference Books:

1. Erwin Kreyszig, Advance Engineering Mathematics, 9th Edition, John Wiley & Sons, 2006.
2. Sheldon Ross, A first course in Probability, 8th Edition, Pearson Education India.
3. W. Feller An Introduction to Probability Theory and its Applications, Vol. 1, 3rd Edition, Wiley, 1968.
4. S. C. Gupta and V. . Kapoor, Fundamentals of Mathematical Statistics, Sultan Chand & Sons.
5. P. G. Hoel, S. C. Port and C. J. Stone, Introduction to Probability Theory, Universal Book Stall, 2003.
6. A. M. Mood, F. A. Graybill and D. C. Bose, Introduction to Theory of Statistics. 3rd Edition, Tata McGraw-Hill Publication.

Thank You