Cauchy's Theorem:

Let G be a finite group and p be a prime such that $p \mid o(G)$, then there exists an element $x \in G$ such that $o(x) = p$.

Proof: Let $o(G) = n$, as G is finite group.

Suppose $p \mid n$, p a prime.

We want to show, that there exists an element $x \in G$ s.t. $o(x) = p$.

We show it by the method of induction on $n = o(G)$.

Case I. Suppose G is an abelian group.

Clearly the result holds for $n = 2$;

As a group of order 2 is cyclic. i.e $G = \langle a \rangle$ and $o(G) = o(a) = 2$.

Now, we suppose that the result is true for all groups having order k less than n.

i.e for $o(G) = k$ and $p \mid k$

If G has only trivial subgroups, then k is prime, hence G is cyclic.

$\therefore$ $G = \langle b \rangle$ s.t. $o(b) = k = p$, a prime.

If G has non-trivial subgroups.

Let $H \leq G$ s.t. $\{e\} \neq H \neq G$.

Since G is abelian. So H is normal in G.

If $p \mid o(H)$,

Since $o(H) < k$.

By our hypothesis, there exists an element $x \in H \subset G$ s.t. $o(x) = p$. ie. $x \in G$ s.t. $o(x) = p$.

If $p \nmid o(H)$, Then Let $o(H) = t$

Since $o(G) = o\left(\frac{G}{H}\right) \cdot o(H)$ and $p \mid o(G) = k$

$\Rightarrow p \mid o\left(\frac{G}{H}\right) < k$.

Also G is abelian implies that $\frac{G}{H}$ is abelian.

So by induction hypothesis, there exists $Hy \in \frac{G}{H}$ s.t. $o(Hy) = p$.

$\Rightarrow (Hy)^p = H$

$\Rightarrow Hy^p = H$

$\Rightarrow y^p \in H \subseteq G$.

$\therefore (y^p)^t = e$ $\quad [\because o(H) = t]$

$\Rightarrow (y^t)^p = e$

$\therefore o(y^t) \mid p \Rightarrow o(y^t) = 1$ or p

If $o(y^t) = 1$ then $y^t = e$

$\Rightarrow Hy^t = H$

$\Rightarrow (Hy)^t = H$

$\Rightarrow o(Hy) \mid t \Rightarrow p \mid t = o(H)$,

a contradiction.

Thus $o(y^t) = p$, $y^t \in G$.

The result holds for $o(G) = k$.

Hence the result is true for all abelian groups.

<u>Case II</u>. Suppose G is any group.

Clearly the result is true for ~~any~~ $n = 1$.

Suppose that the result is true for ally groups having ~~order~~ $k < n$.

Let $o(G) = k$ and $p \mid k$

Let T be a proper subgroup of G

If $p \mid o(T)$, then by Induction hypothesis, there exists $x \in T \subset G$ s.t. $o(x) = p$

i.e $x \in G$ s.t. $o(x) = p$.

If $p \nmid o(T)$, for all $T < G$..

By class equation,

$$|G| = |Z(G)| + \sum_{\substack{i=1 \\ g_i \notin Z(G)}}^{r} |G : C_G(g_i)|$$

$$|G| = |Z(G)| + \sum_{g_i \notin Z(G)} \frac{|G|}{|N(g_i)|}$$

Since $g_i \notin Z(G)$ and $g_i \in N(g_i)$

$\therefore Z(G) < N(g_i) < G$; $\forall i$.

$\Rightarrow p \nmid o(N(g_i))$, $\forall i$

Since $o(G) = o\left(\frac{G}{N(g_i)}\right) \cdot o(N(g_i))$. $\forall i$.

$\Rightarrow p \mid o\left(\frac{G}{N(g_i)}\right)$

By class equation, $p \mid o(Z(G))$.

$\Rightarrow Z(G) = G$ $\{\because p \nmid o(T), \forall T < G\}$

$\Rightarrow G$ is abelian

Since the result is true for all abelian group [by case I)

The result is true for $o(G) = k$.

Hence the result is true for all groups.

Result: An abelian group of order pq, where p and q are distinct, is cyclic.

Solution: Let G be an abelian group and $o(G) = pq$, where p and q are distinct prime.

Clearly $p \mid o(G)$ and $q \mid o(G)$,

So by Cauchy's theorem, there exists $x, y \in G$ s.t. $o(x) = p$ and $o(y) = q$.

Let $H = \langle x \rangle$ and $K = \langle y \rangle$

So $o(H) = p$ and $o(K) = q$

Now $o(xy) = \text{lcm}\{o(x), o(y)\} = pq = o(G)$

$\therefore G = \langle xy \rangle$

Hence G is cyclic.

Problem: Find the class equation of non-abelian group G of order 21.

Solution:- Let $o(G) = 21$, non-abelian,

So $Z(G) < G$.

$\therefore |Z(G)| = 1, 3, 7$.

If $|Z(G)| = 7$ then $\left|\frac{G}{Z(G)}\right| = 3 \Rightarrow \frac{G}{Z(G)}$ is cyclic

$\Rightarrow G$ is abelian, a contradiction.

If $|Z(G)| = 3$, then $\left|\frac{G}{Z(G)}\right| = 7 \Rightarrow \frac{G}{Z(G)}$ is cyclic

$\Rightarrow G$ is abelian, a contradiction.

$\therefore o(Z(G)) = 1$.

ie there is only one conjugacy class of size 1 and rest are of size 3 or 7 as $|cl(x)| \mid |G|$.

So the class equation of G is given by

$$21 = 1 + 3 + 3 + 7 + 7.$$

Theorem: If G is a group of order p^n, p, a prime and $n \geq 1$ and $\{e\} \neq H$ is a normal subgroup of G. Then prove that

$$Z(G) \cap H \neq \{e\}.$$

Proof: Let G be a group of order p^n, p, a prime, $n \geq 1$ and $H \trianglelefteq G$ s.t. $H \neq \{e\}$.

So for $x \in G$,

$$cl(x) \subseteq H \quad \text{or} \quad cl(x) \cap H = \phi.$$

Suppose $cl(x) \cap H \neq \phi$.

Then there exists $y \in G$ such that

$$y \in cl(x) \cap H$$

$$\Rightarrow y \in cl(x) \text{ and } y \in H$$

$$\Rightarrow y = gxg^{-1}, \text{ for some } g \in G$$

And $y \in H \Rightarrow gyg^{-1} \in H$

$$\Rightarrow g(gxg^{-1})g^{-1} \in H$$

$$\Rightarrow g^2 x (g^2)^{-1} \in H, \text{ for } g \in G.$$

$$\Rightarrow g_1 x g_1^{-1} \in H, \text{ for some } g_1 \in G.$$

Let $z \in cl(x) \Rightarrow z = gxg^{-1} \in H$, for some $g \in G$.

$$\Rightarrow z \in H$$

$$\Rightarrow cl(x) \subseteq H$$

Now, By class equation,

$$|G| = |Z(G)| + \sum_{\substack{i=1 \\ g_i \notin Z(G)}}^{r} |G : C_G(g_i)|$$

$$\Rightarrow |H| = |Z(G) \cap H| + \sum_{g_i \in H} |G : C_G(g_i)|$$

Since $o(G) = p^n$, $o(H) > 1$

and $p \mid o(H)$.

Also

$$p \mid |G : C_G(g_i)| \ , \forall i$$

$$\Rightarrow p \mid \sum_{i=1}^{r} |G : C_G(g_i)| \ , \forall$$

Thus $p \mid |Z(G) \cap H| \Rightarrow Z(G) \cap H \neq \{e\}$.

<u>Result:</u> If G is a finite group of order p^n and H is a normal subgroup of order p. Then $H \subseteq Z(G)$.

<u>Solution:</u> Let $o(G) = p^n$ and

$H \trianglelefteq G$ s.t. $o(H) = p$.

By previous result, $H \cap Z(G) \neq \{e\}$.

Also we know that $Z(G) > 1$.

So $H \subseteq Z(G)$.

* If $o(G) = p^n$ and $o(H) = p^2$

Then either $Z(G) \subseteq H$ or $H \subseteq Z(G)$, where H is normal in G.

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