Bounded Linear Transformation on Normed linear Space

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Transformation: Let X and Y be two vector spaces over the same field of scalars. A mapping $T: X \to Y$ is called transformation.

A transformation $T: X \to Y$ is said to be linear if

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) \quad \forall\, x, y \in X,\ \alpha, \beta \in F.$$

Operator: Let X be a vector space over F. A mapping $T: X \to X$ is called an operator. An operator $T: X \to X$ is said to be linear if $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$, $\forall\, x, y \in X$, $\alpha, \beta \in F$.

Bounded linear transformation: Let X and Y be two vector spaces. A linear transformation $T: X \to Y$ is said to be bounded if there exists a real number $M > 0$ such that

$$\|T(x)\| \leq M \|x\| \quad \forall\, x \in X.$$

Such a real number M is said to be a bound for T.

Example: Consider the identity transformation $I: X \to Y$ defined by $I(x) = x$ $\forall\, x \in X$. For all $x \in X$, we have

$$\|I(x)\| = \|x\| \leq K \|x\| \quad \text{where } K \geq 1$$

Hence I is a bounded linear transformation.

Example: The Null transformation $O: X \to Y$ defined by $O(x) = 0$ for all $x \in X$. For all $x \in X$, We have

$$\|O(x)\| = \|0\| = 0 \leq K\|x\|, \text{ where } K > 0.$$

Hence O is a Bounded linear transformation.

<u>Exercise</u> Let T be a linear transform from a vector space X into a vector space Y. Show that

(i) $T(-x) = -T(x) \quad \forall x \in X$

(ii) $T(0) = 0$.

<u>Example</u> Let $(X, \|\ \|)$ be a normed linear space and M be a closed subspace of X. Define T on X into X/M by

$$T(x) = x + M$$

$$\Rightarrow \|T(x)\| = \|x+M\| = \inf_{m \in M} \|x+m\| \leq \|x\| = 1 \cdot \|x\|.$$

Hence T is bounded linear transformation and its bound is 1

<u>Example</u>: Let ℓ^2 be the normed linear space of square summable sequences of complex numbers.

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots)$ be bounded sequence

Define T on $\ell^2 \to \ell^2$ by

$$T(x_1, x_2, \ldots, x_n, \ldots) = (\alpha_1 x_1, \alpha_2 x_2, \ldots, \alpha_n x_n, \ldots)$$

We have

$$\|Tx\|^2 = \sum_{n=1}^{\infty} |\alpha_n x_n|^2$$

$$\leq \sum_{n=1}^{\infty} M^2 |x_n|^2 \quad ; \text{ where } M = \sup_{n \geq 1} |\alpha_n|$$

$$= M^2 \|x\|^2$$

i.e. $\|Tx\| \leq M\|x\| \quad \forall x \in \ell^2$, where $M = \sup_{n \geq 1} |\alpha_n|$

T is bounded linear transformation on ℓ^2.

Example: Consider the Banach space $(C[0,1], \|\ \|_\infty)$. Assume that a function $K: [0,1]\times[0,1] \to \mathbb{R}$ is continuous. Define $T: C[0,1] \to C[0,1]$ by

$$(Tx)(t) = \int_0^1 K(t,\tau)\, x(\tau)\, d\tau$$

$x \in C[0,1]$. Then T is a linear and bounded operator.

Proof: $T(\alpha x+\beta y)(t) = \int_0^1 K(t,\tau)(\alpha x+\beta y)(\tau)\, d\tau$

$$= \int_0^1 K(t,\tau)\left(\alpha x(\tau)+\beta y(\tau)\right) d\tau$$

$$= \alpha \int_0^1 K(t,\tau)\, x(\tau)\, d\tau + \beta \int_0^1 K(t,\tau)\, y(\tau)\, d\tau$$

$$= \alpha T(x)(t) + \beta T(y)(t)$$

Therefore, T is linear.

Now we prove that T is bounded.

Since K is continuous therefore, there exists a constant $C \geq 0$ such that

$$|K(t,\tau)| \leq C \quad \forall\ (t,\tau) \in [0,1]\times[0,1] \quad \cdots (1)$$

Also, since $x \in C[0,1]$, it follows that

$$|x(t)| \leq \sup_{t\in[0,1]} |x(t)| = \|x\|_\infty \quad \cdots (2)$$

Therefore, by using (1) and (2), we get

$$\|Tx\|_\infty = \sup_{t\in[0,1]} |(Tx)(t)|$$

$$= \sup_{t\in[0,1]} \left| \int_0^1 K(t,\tau)\, x(\tau)\, d\tau \right|$$

$$\leq \sup_{t \in [0,1]} \int_0^1 |K(t,\tau)|\, |x(\tau)|\, d\tau$$

$$\leq C\, \|x\|_\infty$$

Thus, T is bounded.

Example: Let $R[x]$ be the set of polynomial. Define norm on $R[x]$ by

$$\|f(x)\| = \sup_{x \in [0,1]} |f(x)| \quad \forall\, f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \in R[x].$$

Define T on $R[x] \to R[x]$ by $T(f(x)) = f'(x)$.

Then T is a linear transformation on $R[x]$.

Let $f_n(x) = x^n \quad \forall\, n \geq 1$

Then $\|f_n\|_\infty = \sup_{x \in [0,1]} |f_n(x)| = \sup_{x \in [0,1]} |x^n| = 1 \quad \forall\, n \geq 1$

and $\|Tf_n\|_\infty = \|n x^{n-1}\|_\infty = n \|x^{n-1}\|_\infty = n \cdot 1 = n$.

i.e. $\|Tf_n\|_\infty = n \|f_n\|_\infty \quad \forall\, n \geq 1$.

Then, there is no fixed real number $M > 0$ such that $\|Tf_n\|_\infty \leq M \|f_n\|_\infty$.

Hence, T is an unbounded linear transformation on $R[x]$.

Definition: Let X and Y be normed linear spaces over the same field F. A transform $T: X \to Y$ (linear or not) is said to be continuous at a point $x_0 \in X$ if there exists a sequence $\{x_n\}$ in X such that

$$x_n \to x_0 \text{ in } X \implies Tx_n \to Tx_0 \text{ in } Y.$$

Equivalently, $T: X \to Y$ is continuous at a point $x_0 \in X$, if for a given $\epsilon > 0$, there exists a real number $\delta > 0$ such that

$$x \in X, \quad \|x - x_0\|_X < \delta \implies \|Tx - Tx_0\|_Y < \epsilon.$$

Further, a transformation $T: X \to Y$ is said to be continuous on X if it is continuous at each $x \in X$.

Theorem. Let X and Y be normed linear spaces over the field F and $T : X \to Y$ a linear transformation. Then, T is continuous on X if and only if T is continuous at a point (any) in X.

Proof: Let T be continuous at a point $x_0 \in X$ and $x \in X$ be arbitrary point of X. Let $\{x_n\} \subset X$ and $x_n \to x$ in X. Then, $\{x_n - x + x_0\}$ is a sequence in X such that $x_n - x + x_0 \to x_0$. Therefore $T(x_n - x - x_0) \to Tx_0 \Rightarrow Tx_n \to Tx$
Hence, T is continuous at x.
The converse statement is obvious.

Theorem: Let X and Y be normed linear spaces over the field F. and $T : X \to Y$ be a linear transformation. Prove that T is uniform continuous if and only if T is bounded.

Proof Assume that T is a bounded linear transformation on X into Y. Then there is a real number $M > 0$ such that

$$\|Tx\| \leq M\|x\| \quad \forall x \in X.$$

Now
$$\|Tx - Ty\| = \|T(x-y)\| \leq M\|x-y\| \quad \forall x, y \in X$$
$$< \epsilon \quad \text{whenever } \|x-y\| < \frac{\epsilon}{M} = \delta$$

i.e. $\|Tx - Ty\| < \epsilon$, whenever $\|x-y\| < \delta = \frac{\epsilon}{M}$.

Therefore T is uniformly continuous on X into Y.
Conversely, ~~Now,~~ let T be uniformly continuous on X, then T is continuous at every point in X. Suppose that T is continuous at some point x_0 in X. Then for $\epsilon = 1 > 0$ there is a $\delta > 0$ such that

$$\|Tx - Tx_0\| < \epsilon \quad \text{whenever } \|x - x_0\| < \delta \quad --(1)$$

Let x be any non-zero vector in X. We have

$$\left\| \left(\frac{\delta}{2} \frac{x}{\|x\|} + x_0\right) - x_0 \right\| = \left\| \frac{\delta}{2} \frac{x}{\|x\|} \right\| = \frac{\delta}{2} < \delta$$

From (1) we get

$$\left\| T\left(\frac{\delta}{2} \frac{x}{\|x\|} + x_0\right) - Tx_0 \right\| < \epsilon$$

$$\Rightarrow \left\| T\left(\frac{\delta}{2} \frac{x}{\|x\|}\right) \right\| < \epsilon$$

$$\Rightarrow \left\| \frac{\delta}{2\|x\|} Tx \right\| < \epsilon$$

$$\Rightarrow \|Tx\| < \frac{2\epsilon}{\delta} \|x\|$$

$$\Rightarrow \|Tx\| < M\|x\| \text{, where } M = \frac{2\epsilon}{\delta}$$

Thus T is a bounded linear transformation on X into Y.

Theorem: Let X and Y be normed linear spaces over the field F and $T: X \to Y$ a linear transformation. Then, T is bounded if and only if T maps bounded sets in X into bounded sets in Y.

Proof: Suppose T is bounded, then ~~$\|Tx\| \leq M\|x\|$~~

$$\|T(x)\| \leq M\|x\|, \quad \forall x \in X \quad \cdots (1)$$

Let S be any bounded subset of X then there exists a constant $K > 0$ such that

$$\|x\| \leq K \quad \forall x \in S \quad \cdots (2)$$

Thus from (1) and (2) we get

$$\|T(x)\| \leq MK \quad \forall x \in S$$

$\Rightarrow \{T(x) : x \in S\}$ is bounded in Y.

Hence, $T(S)$ is bounded set in Y.

Conversely, let S be a closed unit sphere i.e.

$$S = \{x \in X : \|x\| \leq 1\}$$

Then $T(S)$ be bounded so that there exists a real number $M > 0$ such that

$$\|T(x)\| \leq M \text{ for } x \in S$$

If $x = 0$, then $T(x) = 0$ and so $\|T(x)\| \leq M\|x\|$.
If $x \neq 0$, then $\frac{x}{\|x\|} \in S$ and so

$$\left\|T\left(\frac{x}{\|x\|}\right)\right\| \leq M$$

$$\Rightarrow \|T(x)\| \leq M\|x\|$$

Thus T is bounded.

Theorem: Let X and Y be normed linear spaces and T a linear transformation on X into Y. Then the following statements are equivalent

(i) T is continuous at the origin

(ii) T is continuous on X

(iii) T is bounded

(iv) If $S = \{x : \|x\| \leq 1\}$ is closed unit sphere in X, then its image $T(S)$ is a bounded set in Y.

Proof: The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are clear by previous theorems.

Now assume (iv) holds. Therefore there exists a real number $M > 0$ such that

$$\|Tx\| \leq M \quad \text{for } x \in S$$

If $0 = x \in X$, then $Tx = 0$ and so $\|Tx\| \leq M\|x\|$.

If $0 \neq x \in X$, then $\frac{x}{\|x\|} \in S$ and so

$$\left\|T\left(\frac{x}{\|x\|}\right)\right\| \leq M$$

$$\Rightarrow \|Tx\| \leq M\|x\| \quad \forall\ x \in X$$

$$\Rightarrow \|Tx\| < \epsilon, \text{ whenever } \|x\| < \frac{\epsilon}{M} = \delta.$$

Therefore T is continuous at the origin.

Definition: Let T be a bounded linear transformation on X into Y. Then the norm of T is defined as

$$\|T\| = \sup\left\{\frac{\|Tx\|}{\|x\|} ; x \in X \text{ and } x \neq 0\right\}$$

Theorem: Let X and Y be normed linear spaces and $T : X \to Y$ be a linear transformation. Then the following are equivalent.

(a) $\|T\| = \sup\{\|Tx\| ; x \in X ; \|x\| \leq 1\}$

(b) $\|T\| = \sup\{\|Tx\| ; x \in X, \|x\| = 1\}$

(c) $\|T\| = \sup\left\{\frac{\|Tx\|}{\|x\|} ; x \in X, x \neq 0\right\}$

(d) $\|T\| = \inf\{M ; \|Tx\| \leq M\|x\|, x \in X\}$

Proof: Assume that (a) gives the definition of $\|T\|$. If we denote right sides of the expressions in (b), (c) and (d), respectively, by M_2, M_3 and M_4, then it is enough to prove that

$$\|T\| = M_2 = M_3 = M_4.$$

Since $\{x \in X : \|x\| = 1\} \subset \{x \in X : \|x\| \leq 1\}$, it follows that

$$\sup\{\|Tx\| ; x \in X, \|x\| = 1\} \leq \sup\{\|Tx\| ; x \in X, \|x\| \leq 1\}$$

Thus $\|T\| \geq M_2$ - - (1)

Again, since T is a linear transformation, we can write

$$M_3 = \sup\left\{\left\|T\left(\frac{x}{\|x\|}\right)\right\| ; x \in X, x \neq 0\right\}.$$

If $y = \frac{x}{\|x\|}$, then $\|y\| = 1$. Therefore, it implies that

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$M_3 = \sup \{ \|Ty\| ; y \in X, \|y\| = 1 \} = M_2$

Thus, $M_3 = M_2$ . . . (2)

By the definition of M_3, we have

$$M_3 \geq \frac{\|Tx\|}{\|x\|} ; \forall x \in X, x \neq 0$$

$\Rightarrow \|Tx\| \leq M_3 \|x\| \quad \forall x \in X$

Therefore, $M_4 \leq M_3$ . . (3)

Finally, from the definition of M_4, it follows that

$$\|Tx\| \leq M_4 \|x\| \quad \forall x \in X \text{ -}$$

$\Rightarrow \|Tx\| \leq M_4 , \forall x \in X$ with $\|x\| \leq 1$

Therefore, $\|T\| \leq M_4$ - - (4)

From (1), (2), (3) and (4) we get

$$\|T\| \geq M_2 = M_3 \geq M_4 \geq \|T\|$$

$\Rightarrow \|T\| = M_2 = M_3 = M_4$.

<u>Theorem</u>. Let X and Y be normed linear spaces and let T be linear transformation of X into Y. If $T(X)$ is the range of T, then the inverse T^{-1} exists and is bounded (continuous) on $T(X)$ ~~in its domain of definition~~ if and only if there exists a constant $K > 0$ such that

$$K\|x\| \leq \|T(x)\| \quad \text{for all } x \in X. \quad \text{--- (1)}$$

<u>Proof</u>. Let (1) hold and show that T^{-1} exists and continuous. Let $x_1, x_2 \in X$. Then

$$T(x_1) = T(x_2)$$

$$\Rightarrow T(x_1) - T(x_2) = 0$$

$$\Rightarrow T(x_1 - x_2) = 0$$

$$\Rightarrow (x_1 - x_2) = 0 \quad \text{by (1)}$$

$$\Rightarrow x_1 = x_2$$

Hence T is one to one onto $T(X)$, range of T. Therefore T^{-1} exists on $T(X)$. It is easy to prove that T^{-1} is linear. Therefore to each $y \in T(X)$, there exists $x \in X$ such that

$$Tx = y \Leftrightarrow T^{-1}(y) = x \quad \text{--- (2)}$$

Using (2) in (1), we get

$$K\|T^{-1}(y)\| \leq \|y\|$$

$$\Rightarrow \|T^{-1}y\| \leq \frac{1}{K}\|y\| \quad \text{for all } y \in T(X).$$

Hence T^{-1} is bounded and consequently T^{-1} is continuous.

<u>Conversely</u>, let T^{-1} exists and continuous on $T(X)$. Let x be an arbitrary element in X. Since T^{-1} exists there is $y \in T(X)$ such that

$$T^{-1}(y) = x \Leftrightarrow T(x) = y \quad \text{--- (3)}$$

Again since T^{-1} is continuous; it is bounded so that there exists a positive constant M such that

$$\|T^{-1}(y)\| \leq M\|y\|$$

$$\Rightarrow \|x\| \leq M\|T(x)\| \quad \text{by (3)}$$

$$\Rightarrow k\|x\| \leq \|Tx\| \quad \text{where } k = \frac{1}{M} > 0$$

This completes the proof of the theorem.

Theorem: Let X and Y be normed linear spaces and let $\mathcal{B}(X,Y)$ denote the set of all bounded (or continuous) linear transformations from X into Y. Then $\mathcal{B}(X,Y)$ is itself a normed linear space with respect to point wise linear operations

$$(T+U)(x) = T(x) + U(x)$$

$$(\alpha T)(x) = \alpha T(x)$$

and the norm defined by

$$\|T\| = \sup\{\|Tx\| : x \in X,\ \|x\| \leq 1\}$$

Furthermore, if Y is a Banach space then $\mathcal{B}(X,Y)$ is also Banach space.

Proof: It is easy to prove that $\mathcal{B}(X,Y)$ is a linear space.

[It is well-know that the set L of all linear transformations from one linear space into another linear space is itself a linear space with respect to the pointwise linear operations. Therefore in order to prove that $\mathcal{B}(X,Y)$ is a linear space, it is sufficient to show that $\mathcal{B}(X,Y)$ is a subspace of L. Let $T_1, T_2 \in \mathcal{B}(X,Y)$, then there exist real numbers $K_1 > 0$ and $K_2 > 0$ such that

$$\|T_1(x)\| \leq K_1 \|x\| \text{ and } \|T_2(x)\| \leq K_2 \|x\|$$

for all $x \in X$.

If α, β are any scalars, then

$$\begin{aligned}
\|(\alpha T_1 + \beta T_2)(x)\| &= \|(\alpha T_1)(x) + (\beta T_2)(x)\| \\
&= \|\alpha T_1(x) + \beta T_2(x)\| \\
&\leq \|\alpha T_1(x)\| + \|\beta T_2(x)\| \\
&\leq (|\alpha| K_1 + |\beta| K_2)\|x\|
\end{aligned}$$

i.e. $\|(\alpha T_1+\beta T_2)(x)\| \leq M\|x\|$, where $M=(|\alpha|K_1+|\beta|K_2)$

i.e. $\alpha T_1+\beta T_2 \in \mathcal{B}(X,Y)$. Hence $\mathcal{B}(X,Y)$ is also a linear space.]

Now we verify the norm postulates

(i) Clearly $\|T\| = \sup\{\|Tx\| : x\in X, \|x\|\leq 1\} \geq 0$ $\quad \sup\left\{\frac{\|Tx\|}{\|x\|} : x\in X, x\neq 0\right\}$

(ii) $\|T\|=0 \iff \sup\{\|Tx\| : x\in X, \|x\|\leq 1\}=0$

$\iff \|Tx\|=0$ for all $x\in X$

$\iff Tx=0$ for all $x\in X$

$\iff T=O$ (Null transformation)

(iii) Let $T_1, T_2 \in \mathcal{B}(X,Y)$, then

$\|T_1+T_2\| = \sup\{\|(T_1+T_2)(x)\| ; x\in X, \|x\|\leq 1\}$

$= \sup\{\|T_1(x)+T_2(x)\| ; x\in X, \|x\|\leq 1\}$

$\leq \sup\{\|T_1(x)\|+\|T_2(x)\| ; x\in X, \|x\|\leq 1\}$

$= \|T_1\|+\|T_2\|$

i.e. $\|T_1+T_2\| \leq \|T_1\|+\|T_2\|$.

(iv) Let α be any scalar, then

$\|\alpha T\| = \sup\{\|(\alpha T)(x)\| : x\in X, \|x\|\leq 1\}$

$= \sup\{|\alpha|\,\|T(x)\| : x\in X, \|x\|\leq 1\}$

$= |\alpha| \sup\{\|T(x)\| : x\in X, \|x\|\leq 1\}$

$= |\alpha|\,\|T\|$

Thus $\mathcal{B}(X,Y)$ is a normed linear space.

Now we show that $B(X,Y)$ is complete if Y is complete.

Let $\{T_n\}$ be a Cauchy sequence in $B(X,Y)$.

Then $\|T_m - T_n\| \to 0$ as $m,n \to \infty$ - - (1)

Then, for each $x \in X$, we have

$$\|T_m(x) - T_n(x)\| = \|(T_m - T_n)(x)\|$$
$$\leq \|T_m - T_n\| \, \|x\| \quad (\because T_m - T_n \text{ is bounded})$$

i.e. $\|T_m(x) - T_n(x)\| \to 0$ as $m,n \to \infty$ - - (2)

$\Rightarrow$ $\{T_n(x)\}$ is a Cauchy-sequence in Y for each $x \in X$. Since Y is complete, there exists a vector $T(x)$ in Y such that $T_n(x) \to Tx$.

$$\text{Since } T(\alpha x + \beta y) = \lim_{n\to\infty} T_n(\alpha x + \beta y)$$
$$= \lim_{n\to\infty} T_n(\alpha x) + \lim_{n\to\infty} T_n(\beta y)$$
$$= \alpha \lim_{n\to\infty} T_n(x) + \beta \lim_{n\to\infty} T_n(y)$$
$$= \alpha T(x) + \beta T(y)$$

Therefore T is linear.

Now, we show that T is bounded.

Since $\big|\, \|T_m\| - \|T_n\| \,\big| \leq \|T_m - T_n\| \to 0$ as $m,n \to \infty$ by (1)

$\Rightarrow$ $\{\|T_n\|\}$ is Cauchy sequence in $\mathbb{R}$.

$\Rightarrow$ $\{\|T_n\|\}$ is Convergent (Since $\mathbb{R}$ is complete)

$\Rightarrow$ $\{\|T_n\|\}$ is bounded (Since every Convergent sequence is bounded)

$\Rightarrow$ $\exists$ a real number M such that

$\|T_n\| \leq M \quad \forall n.$

Now

$$\|T(x)\| = \|\lim_{n\to\infty} T_n(x)\|$$

$$= \lim_{n\to\infty} \|T_n(x)\| \quad \text{(Since } \| \; \| \text{ is continuous function)}$$

$$\leq \lim_{n\to\infty} \|T_n\| \, \|x\|$$

$$\leq \lim_{n\to\infty} M \, \|x\|$$

$$\leq M \|x\|$$

Therefore, T is bounded, thus $T \in \mathcal{B}(X, Y)$.

<u>Claim</u> $T_n \to T$

Since $\{T_n\}$ is a Cauchy sequence in $\mathcal{B}(X, Y)$ for any given $\epsilon > 0$, there is a positive integer N such that

$$m, n \geq N \Rightarrow \|T_n - T_m\| < \epsilon.$$

For an arbitrary $x \in X$ with $\|x\| \leq 1$ and for $m, n \geq N$,

$$\|T_n x - T_m x\| = \|(T_n - T_m)(x)\| \leq \|T_n - T_m\| \, \|x\| \leq \|T_n - T_m\| < \epsilon.$$

Letting $m \to \infty$ we get $\|T_n x - Tx\| \leq \epsilon \quad \forall \, n \geq N$

Hence $\sup_{\|x\| \leq 1} \|T_n(x) - Tx\| \leq \epsilon$

This means that $\|T_n - T\| \leq \epsilon \quad \forall \, n \geq N$.

Thus, $\{T_n\}$ converges to T and $T \in \mathcal{B}(X, Y)$. Therefore, $\mathcal{B}(X, Y)$ is a complete normed linear space.

Hence $\mathcal{B}(X, Y)$ is a Banach space.

Definition: The null space of a linear transformation $T: X \to Y$, where X and Y are linear spaces, is defined by

$$\mathcal{N}(T) = \{x \in X : Tx = 0\}$$

where 0 is the zero vector of Y and $\mathcal{N}(T)$ is a subspace of X.

Theorem: Let X and Y be normed linear spaces over the field F and $T: X \to Y$ a bounded (continuous) linear operator. Then, the null space $\mathcal{N}(T)$ is closed.

Proof: Let $x \in \overline{\mathcal{N}(T)}$ be arbitrary. Then, $\exists$ a sequence $\{x_n\} \subset \mathcal{N}(T)$ such that $x_n \to x$. Since T is continuous, $Tx_n \to Tx$. But $x_n \in \mathcal{N}(T)$. Thus $x \in \overline{\mathcal{N}(T)} \Rightarrow x \in \mathcal{N}(T)$, i.e. $\overline{\mathcal{N}(T)} \subseteq \mathcal{N}(T)$. But we always have $\mathcal{N}(T) \subset \overline{\mathcal{N}(T)}$ and so finally we have $\mathcal{N}(T) = \overline{\mathcal{N}(T)}$ and hence $\mathcal{N}(T)$ is closed.

Remark The range of a bounded linear operator need not be closed.

Reference Books:

1. E. Kreyszig, Introductory Functional Analysis with Applications, John Wiley & Sons, 2006.
2. Walter Rudin, Functional Analysis, Tata McGraw Hill, 2010.
3. G. Bachman and L. Narici, Functional Analysis, Dover Publications, 2000.
4. J. B. Conway, A Course in Functional Analysis, Springer, 2006
5. P.K. Jain, O.P. Ahuja and K. Ahmed, Functional Analysis, New Age International.
6. B.V. Limaye, Functional Analysis, New Age International.
7. A. E. Taylor, Introduction to Functional Analysis, John Wiley, 1958.
8. G.F. Simmons, Introduction to Topology and Modern Analysis, McGraw Hills, 1963.