PHYS 3014: Statistical Mechanics

Lecture Notes
Part 12

# Maxwell-Boltzmann, Bose-Einstein and Fermi-Dirac Statistics 1


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Programme: B. Sc. Physics
Semester: VI

# Identical Particles and Symmetry Requirements

Consider a system of N identical structureless particles in a container of volume V. Let $Q_i$ denotes collectively all the coordinates of ith particle. Let $s_i$ be the index representing a possible quantum state of this single particle. Then the state of the system is described by set of quantum numbers $\{s_1, s_2 \cdots s_N\}$. which characterizes the wavefunction $\psi$ of the system in this state.

$$\psi = \psi_{\{s_1, s_2 \cdots s_N\}}(Q_1, Q_2, \cdots Q_N)$$

## Classical Case (Maxwell-Boltzmann Statistics)

In this case particles are considered to be distinguishable and any number of particles can be in the same single particle state s. When two particles are interchanged, there is no symmetry requirements on the wavefunction. We will get completely a new state of the system. The particles are then said to obey Maxwell-Boltzmann statistics.

## Quantum Case

There must be symmetry requirements on the wavefunction under interchange of any two identical particles. We do not get any new state of the system on interchange of particles. For counting the distinct possible states accessible to the system, the particles must be considered as indistingu-ishable. There are two possible cases

(a) Particles with integral spin (Bose-Einstein Statistics)

The symmetry requirement is that the total wavefunction $\psi$ should be symmetric (remain unchanged) under interchange of any two particles.

$$\psi(\cdots Q_j \cdots Q_i \cdots) = \psi(\cdots Q_i \cdots Q_j \cdots)$$

The interchange of two particles does not lead to a new state of the system. For counting the possible states accessible to the system, particles are considered as indistinguishable. There is no restriction on number of particles accomodating in any one single particle states. Particles are said to obey Bose-Einstein Statistics and are called as Bosons (ex. $He^4$ atoms, photons etc). Phonons

(b) Particles with half integral spin (Fermi-Dirac Statistics)

The symmetry requirement is that the total wavefunction $\psi$ must be antisymmetric (it changes sign) under interchange of any two particles.

$$\psi(\cdots Q_j \cdots Q_i \cdots) = -\psi(\cdots Q_i \cdots Q_j \cdots)$$

The Interchange of two particles does not lead to a new state of the system. For finding the possible states accessible to the system, particles are considered as indistinguishable.

Now suppose that two particles are in the same single particle state s are interchanged then we have

$$\psi(\cdots Q_j \cdots Q_i \cdots) = \psi(\cdots Q_i \cdots Q_j \cdots)$$

But due to symmetry requirement, the two conditions must be valid when $\psi = 0$ i.e when $i^{th}$ and $j^{th}$ particles are in the same state s.

It implies that two or more particles can not accomodate to the same single particle state. The fact that no two fermions are allowed to occupy the same state is known as Pauli's exclusion principle. Particles satisfying antisymmetric requirements are said to obey Fermi-Dirac statistics and are called as Fermions (Ex. $He^3$ atoms, electrons, neutrons etc).

## Ideal Gas in microcanonical Ensemble

In ideal gas, the interaction between the particles is negligible. So the energy of the system is the sum of the kinetic energy of each particle.

An ideal gas system may be of three types

(i) Maxwell-Boltzmann system    Classical statistics

- particles are distinguishable
- any number of particles can accomodate a single particle state

(ii) Bose-Einstein System    Quantum statistics

- particles are indistinguishable
- any number of particles can accomodate a single particle state

(iii) Fermi-Dirac System Quantum Statistics

- particles are indistinguishable
- only one particle can accomodate a single particle state

Consider a system of N non-interacting particles confined in a volume V and sharing an energy E. The microcanonical macrostate is represented by (N, V, E). We have to determine total no. of microstates $\Omega(N,V,E)$ accessible to the system under the macrostate (N, V, E).

For large V, the single particle energy levels in the system are very close to one another. So we can divide the energy spectrum into a large number of groups of levels which are referred to as energy cell.

Let $\epsilon_i$ are the average energy of $i^{th}$ energy cell and $g_i$ is the degeneracy

(no. of levels) of $i^{th}$ energy cell. For all $i$, $g_i \gg 1$.
$n_i$ be the occupation number or number of particles in the $i^{th}$ cell. So the distribution set $\{n_i\}$ must fulfil the conditions

$$\sum_i n_i = N$$

and $$\sum_i n_i \epsilon_i = E$$

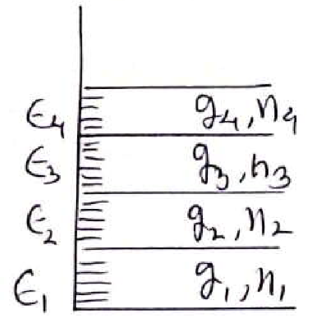
$\therefore$ number of microstates

$$\Omega(N, V, E) = \sum_{\{n_i\}}{}' W\{n_i\}$$

where $W\{n_i\}$ is the number of distinct microstates associated with the distribution set $\{n_i\}$ and $\Sigma'$ goes over all the distribution sets following the given conditions.

Now $$W\{n_i\} = \prod_i \omega(i)$$

where $\omega(i)$ is the number of distinct microstates associated with the $i^{th}$ cell of the energy spectrum. The $i^{th}$ cell containing $n_i$ particles accomodating among $g_i$ levels. So $\omega(i)$ is the number of ways in which $n_i$ identical particles can be distributed among $g_i$ levels of the $i^{th}$ cell.




Maxwell - Boltzmann Case (distinguishable Particles) classical particle

N identical and distinguishable particles are distributed among different energy cell. $\epsilon_i$ having degeneracy $g_i$. $n_i$ particles are accomodating in $\epsilon_i$. such that

$$E = \sum_i n_i \epsilon_i \quad \text{and} \quad \sum_i n_i = N$$

The number of ways of distributing N distinguishable particles such that $n_1$ particles in the energy cell of energy $\epsilon_1$, $n_2$ particles in the energy cell of energy $\epsilon_2$ etc. will be

$$^{N}C_{n_1} \cdot {}^{N-n_1}C_{n_2} \cdot {}^{N-n_1-n_2}C_{n_3} \cdots$$

$$= \frac{\lfloor N}{\lfloor n_1 \lfloor N-n_1} \times \frac{\lfloor N-n_1}{\lfloor n_2 \lfloor N-n_1-n_2} \times \cdots$$

$$= \frac{\lfloor N}{\lfloor n_1 \lfloor n_2 \lfloor n_3 \cdots}$$

$$= \frac{\lfloor N}{\prod_i \lfloor n_i}$$

This could also be thought as follows.

N particles are arranged in $\lfloor N$ ways and permutation of $n_i$ particles among their respective energy cell of energy $\epsilon_i$ will not alter the groupings. So total no. of ways will be $\frac{\lfloor N}{\lfloor n_1 \lfloor n_2 \cdots}$

$$= \frac{\lfloor N}{\prod_i \lfloor n_i}$$

Within an energy cell of energy $\epsilon_i$, the probability of occupancy of a state by a particle is $g_i$ then by principle of equal a priory probability for each state, the probability of occupancy of level $\epsilon_i$ by $n_i$ particles is $g_i^{n_i}$.

Therefore, the total number of possible ways for occupancy in all energy cells is

$$W\{n_i\} = \frac{\lfloor N}{\lfloor n_1 \lfloor n_2 \lfloor n_3 \cdots} \cdot (g_1)^{n_1} (g_2)^{n_2} \cdots$$

$$= \lfloor N \prod_i \frac{(g_i)^{n_i}}{\lfloor n_i}$$

Incorporating the Gibbs correction factor, the number of distinct ways i.e number of distinct microstates associated with the distribution $\{n_i\}$

$$W_{M.B}\{n_i\} = \prod_i \frac{(g_i)^{n_i}}{\lfloor n_i}$$

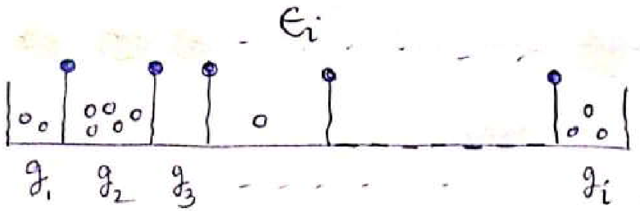
Bose-Einstein Case (Indistinguishable particles) Bosons

N identical and indistinguishable particles are distributed among different energy cells of energy $\epsilon_i$ each having $g_i$ levels in such a way that any number of particles can accomodate a single particle state. We have

$$\sum_i n_i = N$$

and $$\sum_i n_i \epsilon_i = E$$

$n_i$ is the number of particles accomodating in energy cell of energy $\epsilon_i$.




The energy cell of energy $\epsilon_i$ could be partitioned into $g_i$ boxes by $g_i - 1$ partitions. Now we have to distribute $n_i$ identical indistinguishable particles among the $g_i$ boxes subjected to the condition that any box can have any no. of particles. The no of ways for such arrangement is

$$(n_i + g_i - 1)!$$

Particles are indistinguishable so permutations $n_i!$ among particles as well as permutations $(g_i - 1)!$ among partitions do not change the distribution. Therefore number of distinct distribution of bosons in the $i^{th}$ cell of the energy spectrum is

$$W(i) = \frac{(n_i + g_i - 1)!}{n_i! \, (g_i - 1)!}$$

Therefore, total number of ways to distribute

N identical and indistinguishable particles among all energy cell associated with the distribution set $\{n_i\}$

$$W_{B.E}\{n_i\} = \prod_i \frac{\lfloor n_i + g_i - 1}{\lfloor n_i \lfloor g_i - 1}$$

Fermi-Dirac Case (indistinguishable particles) Fermions

N identical and indistinguishable particles are distributed among different energy cells energy $\epsilon_i$ having energy levels $g_i$ in such a way that each energy level can accomodate at most one particle. subjected to condition that

$$\sum_i n_i = N \quad \text{and} \quad \sum_i n_i \epsilon_i = N$$

If $\epsilon_i$ energy cell accomodate $n_i$ particles than $n_i \leq g_i$ no. of energy levels in the cell.

So we have to place $n_i$ particles among $g_i$ energy levels. The first particle will be put in any one of $g_i$ state in $g_i$ different ways. The second particle will be placed any one of the remaining $g_i - 1$ states in $g_i - 1$ different ways. Proceeding with the same way the last particle will be placed in $(g_i - n_i + 1)$ ways.

Therefore total no of ways to distribute $n_i$ particles among $g_i$ levels is

$$= g_i \cdot (g_i - 1) \cdot (g_i - 2) \cdot \cdots (g_i - n_i + 1)$$

$$= \frac{\lfloor g_i}{\lfloor g_i - n_i}$$

Since the particles are indistinguishable so $\lfloor n_i$ ways of permutations of $n_i$ particles one in each state does not give any new arrangement.

Therefore, the number of distinct ways to distribute $n_i$ particles in energy cell of energy $\epsilon_i$ is

$$\omega(i) = \frac{\lfloor g_i}{\lfloor n_i \, \lfloor g_i - n_i}$$

This can also be thought as to find the number of ways in which $g_i$ levels can be divided into two subgroups. One consisting of $n_i$ levels which are filled with one particle each and second consisting of $(g_i - n_i)$ levels which are unoccupied. Such number of ways is $\frac{\lfloor g_i}{\lfloor n_i \, \lfloor g_i - n_i}$

Therefore, total number of ways to distribute N identical particles among all energy cells associated with distribution set $\{n_i\}$

$$W_{FD}\{n_i\} = \prod_i \frac{\lfloor g_i}{\lfloor n_i \, \lfloor g_i - n_i}$$

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# Thank You

**For any questions/doubts/suggestions and submission of assignments**

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