DIFFERENT TYPES OF SINGULARITIES OF AN ANALYTIC FUNCTION

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ZEROS OF AN ANALYTIC FUNCTION

Let $f(z)$ be analytic in a domain D and let a be ~~th~~ a point of D. Then $f(z)$ can be expanded as Taylor's series about $z=a$ in the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n,$$

If $a_0 = a_1 = \cdots = a_{m-1} = 0$ and $a_m \neq 0$, we say that $f(z)$ has a zero of order m at $z=a$.

A zero of order one is said to be simple zero.

when $f(z)$ has a zero of order "m" at $z=a$.

Taylor's expansion reduces to

$$f(z) = \sum_{n=m}^{\infty} a_n (z-a)^n = \sum_{n=0}^{\infty} a_{n+m} (z-a)^{m+n}$$

$$= (z-a)^m \underbrace{\sum_{n=0}^{\infty} a_{n+m} (z-a)^n}_{g(z)}$$

$f(z) = (z-a)^m g(z)$ where g is analytic and $g(a) \neq 0$

THEOREM:

Every zero of an analytic function f not equal to zero is isolated.

- i.e.

Let $f(z)$ be analytic in a domain D, then unless $f(z)$ is identically zero, there exist a neighbourhood of each point in D throughout which the function has no zero, except possibly at the point itself.

Proof. Let $f(z)$ has a zero of order m at a.
then $\exists$ an $R > 0$ s.t.

$$f(z) = (z-a)^m g(z), \quad |z-a| < R$$

where g is analytic at a and $g(a) \neq 0$.

Let $|g(a)| = 2\varepsilon > 0$, then as g is contin. at a,
$\exists$ a $\delta > 0$ s.t.

$$|g(z) - g(a)| < \varepsilon \quad \text{whenever} \quad |z-a| < \delta$$

$\therefore$ when $|z-a| < \delta$ we have

$$|g(z)| = |g(a) - [g(a) - g(z)]| \geq |g(a)| - |g(a) - g(z)|$$

$$> 2\varepsilon - \varepsilon = \varepsilon$$

Thus $g(z) \neq 0$ in $|z-a| < \delta$.

$\because (z-a)^m \neq 0$ in $0 < |z-a| < \delta$

Hence, $f(z) = g(z)(z-a)^m \neq 0$ in $0 < |z-a| < \delta$ except at a. $\Rightarrow$ zero is isolated.

Different Types of Singularities

- The points of the domain where the function is not analytic are called singular points.

- Singularities are of different types, depends upon the nature of the function in the neighbourhood of the singular point.

ISOLATED SINGULARITY

① Isolated Singularity z_0 is said to be isolated singularity of a fn $f(z)$, if fn is analytic at each point in some nbd of z_0

Ex. ① $f(z) = \frac{z+3}{z^2(z^2+2)}$ has 3 isolated singular pts $z=0$, $z=\sqrt{2}i$, $z=-\sqrt{2}i$

② $\frac{1}{\sin(\pi/z)}$, infinite number of isolated singularity $z = \pm\frac{1}{n}$ $n=1,2,3,\ldots$

origin is also singularity, but is not isolated

Poles, Isolated Essential Singularity and Removable Singularity

singularity

Let $z = z_0$ be isolated singularity of $f(z)$.

$\Rightarrow \exists$ a deleted nbd $0 < |z - z_0| < r$ where $f(z)$ is analytic. Then $f(z)$ has Laurent's expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \underbrace{\sum_{n=1}^{\infty} b_n (z - z_0)^{-n}}_{\text{Principal part of Laurent series}}$$

POLE

- If principal part contain finite number of terms, say m then the singular point is called pole of order m of f(z).

- A pole of order one is called simple pole.

ISOLATED ESSENTIAL SINGULARITY

If the principal part of contains an infinite no. of terms, then z_0 is called an isolated essential singularity of $f(z)$.

Ex. $e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots$ 0 is an essential singularity.

REMOVABLE SINGULARITY

<u>Removable Singularity</u> If Principal part consist of no terms, then z_0 is called removable singularity of $f(z)$.

In this case we can remove the singularity by defining the fns at $z = z_0$. ~~in such a way that~~

<u>Ex.</u> $f(z) = \frac{\sin z}{z}$ has removable singularity at $z = 0$.

we define $f(z) = 1$ at $z = 0$

THEOREM:

A function which has no singularity in the extended complex plane is constant.

Proof. $\because f(z)$ has no singularity

$\Rightarrow$ we can expand $f(z)$ as a Taylor's series in any circle $|z| = R$, R is large.

Thus $f(z) = \sum_{n=0}^{\infty} a_n z^n$ —— (*)

Again since $f(z)$ has no singularity at $z = \infty$,

$\Rightarrow f\left(\frac{1}{z}\right)$ is analytic at $z = 0$.

Taylor's series

By ~~putting~~ replacing z by $\frac{1}{z}$ in * we get

$$f\left(\frac{1}{z}\right) = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots \qquad 0 < |z| \leq \infty \quad \text{—} (**)$$

From (**) we can see that $z = 0$ is an essential singularity, contradicⁿ as $f\left(\frac{1}{z}\right)$ is analytic at $z = 0$.

This is only possible if $a_n = 0$, $n \geq 1$

Thus $f\left(\frac{1}{z}\right) = a_0$. Or $f(z) = a_0 \Rightarrow$ fun is const

RESIDUE AT A POLE

Residue at a pole

Let z_0 be a pole of order m of $f(z)$ so that

$$f(z) = \sum_{n=0}^{\infty} (z-z_0)^n + \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_m}{(z-z_0)^m}$$

Where $b_m \neq 0$. Then the coefficient b_1 (may be zero) is called the residue of $f(z)$ at z_0.

Theorems on Poles and Other Singularities

Th-1 If $f(z)$ has a pole of order m at $z = z_0$, then the fns ϕ defined by

$$\phi(z) = (z - z_0)^m f(z)$$ has a removable singularity at z_0 and that $\phi(z_0) \neq 0$, also the residue at z_0 is given by

$$\frac{\phi^{(m-1)}(z_0)}{(m-1)!}.$$

Proof. Since $f(z)$ has a pole of order m, we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_m}{(z-z_0)^m} \quad \text{--- (I)}$$

when $0 < |z-z_0| < r$ and $b_m \neq 0$.

now, assume

$$\phi(z) = (z-z_0)^m f(z) \quad \text{--- (II)}$$

$\phi(z)$ is not defined at z_0.

using (I) & (II)

$$\phi(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^{m+n} + b_1 (z-z_0)^{m-1} + b_2 (z-z_0)^{m-2} + \cdots + \cdots b_m \quad \text{--- (3)}$$

we now define $\phi(z_0) = b_m$ so that

$\phi(z_0) \neq 0$

$\Rightarrow \phi(z)$ has a removable singularity at $z = z_0$.

$$\phi(z_0) = \lim_{z \to z_0} (z-z_0)^m f(z)$$

coefficient of $(z-z_0)^{m-1}$ is $\dfrac{\phi^{(m-1)}(z_0)}{(m-1)!}$ which is

residue at z_0

\# In particular when z_0 is a simple pole then residue at z_0 is

$$\phi(z_0) = \lim_{z \to z_0} (z-z_0) f(z)$$

THANK YOU !!