Different Types of Singularities of an Analytic Function
continued...

By
Dr. Babita Mishra
Assistant Professor,
Department of Mathematics,
School of Physical Sciences.
Mahatma Gandhi Central University,
Motihari, Bihar.

THEOREM 2

Theorem. Let a function $f(z)$ be analytic in an open domain D and let $\phi(z) = \frac{1}{f(z)}$ where $f(z) \neq 0$. Then f has a zero of order m at z_0 iff ϕ has a pole of order m at z_0.

Proof Let $\Phi(z) = \frac{1}{f(z)}$ has a pole of order m at z_0.
We have to show $f(z)$ has a zero of order m.
We can write $\Phi(z) = \frac{\psi(z)}{(z-z_0)^m}$ where $\psi(z)$ is analytic in some nbd of z_0, and $\psi(z_0) \neq 0$

Hence, $f(z) = \frac{1}{\phi(z)} = \frac{(z-z_0)^m}{\psi(z)}$

$\left\{ \begin{array}{l} \because \psi(z) \text{ is analytic and } \psi(z_0) \neq 0 \\ \Rightarrow \frac{1}{\psi(z)} \text{ is also analytic.} \end{array} \right.$

$\Rightarrow f(z)$ has a zero of order m at z_0

Conversely

Let $f(z)$ has a zero of order m at z_0.

Then $f(z) = (z-z_0)^m \psi(z)$, where $\psi(z_0) \neq 0$

$$\Rightarrow \frac{1}{\psi(z)} = \frac{(z-z_0)^m}{f(z)}$$ is analytic in a nbd of z_0.

So it has Taylor's series expansion

$$\frac{1}{\psi(z)} = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots + a_m(z-z_0)^m + \cdots$$

$$\therefore \phi(z) = \frac{1}{f(z)} = \frac{1}{(z-z_0)^m \psi(z)}$$

$\phi(z)$ has a pole of order m $\Leftarrow$ $$= \frac{a_0}{(z-z_0)^m} + \frac{a_1}{(z-z_0)^{m-1}} + \cdots a_m + \sum_{n=1}^{\infty} a_{m+n}(z-z_0)^n$$

THEOREM 3:
Poles are isolated.

- i.e.

If f(z) has a pole at a point then in the neighbourhood of that point f(z) does not contain any other pole.

Proof. We know that if $f(z)$ has a pole of order m at z_0, then $\exists$ a deleted nbd $0 < |z - z_0| < \delta$ of z_0 in which $f(z)$ is analytic and has a Laurent series expansion of the form.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}$$

$\Rightarrow$ In the nbd of z_0, the only pole is at z_0.

$\Rightarrow$ poles are isolated.

THEOREM 4:

Th// Let z_0 be an isolated singularity of $f(z)$ and if $|f(z)|$ is bounded on some deleted nbd of z_0, ~~in~~ ~~which~~ then z_0 is a removable singularity.

Proof. Given that $|f(z)|$ is bounded on some deleted nbd of z_0.

$\Rightarrow \exists\, M$ (max value of $f(z)$) on a circle γ defined by $|z-z_0| = r$, where r is so chosen that γ lies entirely within deleted nbd.

Then Laurent's th. gives

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \quad \Bigg\{ \text{where } a_n = \frac{1}{2\pi i}\int_{\gamma} \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

now $|a_n| \leq \frac{1}{2\pi} \int_{\gamma} \frac{|f(z)||dz|}{|z-z_0|^{n+1}} \leq \frac{1}{\cancel{2\pi}} \frac{M}{r^{n+1}} \cdot \cancel{2\pi} r$

$$= \frac{M}{r^n} \longrightarrow 0 \text{ as } n \to \infty$$

$\Rightarrow a_n = 0$ for every $(-)$ve values of n so that the Principal part of $f(z)$ doesn't contain any $(-)$ve power of $z - z_0$ in the expansion

$\Rightarrow z_0$ is removable singularity of $f(z)$.

THEOREM 5:

Weierstrass's theorem

Statement:

Let z_0 be essential singularity of a function $f(z)$ and let c be an arbitrary const. Then for every $\varepsilon > 0$ and every nbd $0 < |z - z_0| < \rho$ of z_0, $\exists$ a pt z of this nbd s.t. $|f(z) - c| < \varepsilon$.

i.e.

In every arbitrary neighbourhood of an essential singularity their exist a point (therefore can be infinite number of points) at which the function defers as little as we please from any pre assigned number.

Proof If possible let the th. is not true. Then for a given $\varepsilon > 0$ $\exists$ c and a (+)ve no. ρ $\ni$

$$|f(z)-c| > \varepsilon \text{ for every } z \text{ lying } 0<|z-z_0|<\rho$$

i.e. $$\frac{1}{|f(z)-c|} < \frac{1}{\varepsilon}$$

by previous th. $\frac{1}{f(z)-c}$ has a removable singularity at z_0. So its principal part contains no (−)ve power of $z-z_0$. Thus in the nbd of z_0, we have

$$\frac{1}{f(z)-c} = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

If $a_0 \neq 0$, we define

$$\frac{1}{f(z_0)-c} = a_0 \Rightarrow f(z_0) = c + \frac{1}{a_0}$$

$\Rightarrow \frac{1}{f(z)-c}$ is analytic and non zero for $z = z_0$

$\Rightarrow f(z)$ is analytic at $z = z_0$ {contradiction}

Again, let $a_0 = a_1 = \cdots = a_{m-1} = 0$ & $a_m \neq 0$

$$\frac{1}{f(z)-c} = a_m(z-z_0)^m + a_{m+1}(z-z_0)^{m+1} + \cdots$$

$$=(z-z_0)^m \sum_{n=0}^{\infty} a_{m+n} (z-z_0)^n$$

$\Rightarrow$ $z = z_0$ is a zero of order m of $\frac{1}{f(z)-c}$

$\Rightarrow$ $f(z)-c$ has a pole of order m at z_0.

$\because$ c is a const

$\Rightarrow$ $f(z)$ has a pole of order m at $z = z_0$.

contradiction

$\Rightarrow$ Theorem is true.

LIMIT POINT OF ZEROS

Th// Let $f(z)$ be analytic in D and let E be a set of zeros of $f(z)$ having a limit point α in D. Then $f(z)$ vanishes identically in D.
i.e. $f(z)$ vanishes for all $z \in D$.

Proof - $\because$ α is limit pt. of the set E of zeros of $f(z)$

$f(z)$ vanishes at an infinite no. of pts. in every nbd of α. i.e. $f(z)$ has zeros as near as we please to α.

As $f(z)$ is contin. at α.

We must have $f(\alpha) = 0$

But $\because$ zeros are isolated, α cannot be a zero of $f(z)$ unless $f(z)$ vanishes identically in D.

IDENTITY THEOREM

Statement:

If f(z) and g(z) are analytic in a domain D and if f(z)=g(z) on a subset E of D which has a limit point α in D, then f(z)=g(z) in the whole of D.

Proof. Let $F(z) = f(z) - g(z)$ and $F(z)$ vanishes on E.

$\because \alpha$ is a limit pt. of E.

$\Rightarrow F(z)$ must vanishes at an infinite no. of pts in every nbd of α.

Continuity of $F(z) \Rightarrow F(\alpha) = 0$

But $\because$ zeros are isolated.

α cannot be a zero of $F(z)$ unless $F(z)$ vanishes identically in D.

$\Rightarrow f(z) = g(z)$ in whole of D.

EXAMPLES

<u>Example</u> Classify the nature of the singularity of the fns $f(z) = \dfrac{e^{-z}}{(z-2)^4}$ and compute the residue.

Laurent

Sol^n $\because$ $z = 2$ is the singularity. Write the Laurent series expansion in $0 < |z-2| < R$

$$f(z) = \frac{e^{-2} e^{-(z-2)}}{(z-2)^4} = e^{-2} \left[\frac{1}{(z-2)^4} - \frac{1}{(z-2)^3} + \frac{1}{2!\,(z-2)^2} - \frac{1}{3!\,(z-2)} + \frac{1}{4!} \frac{\cancel{(z-2)^4}}{\cancel{(z-2)^4}} - \frac{1}{5!}(z-2) + \cdots \right]$$

Thus $z = 2$ is a pole of order 4.

Residue is coefficient b_1 of $\frac{1}{z-2}$ which is $-\frac{1}{6} e^{-2}$

EXAMPLE 2:

The zeros of the fns $\sin\frac{1}{z}$ are given by

$$z = \pm\frac{1}{n\pi} \quad n = 1, 2, \ldots$$

limit pt. of these zeros is the pt. $z = 0$.
Thus 0 is an isolated singularity of $\sin\frac{1}{z}$.

explain

Again the fns $\tan\frac{1}{z}$ has poles at pt. given by

$$z = \frac{2}{n\pi} \quad n = \pm 1, \pm 3, \ldots$$

limit pt. of this sequence of poles is $z = 0$ which is non-isolated essential singularity.

EXAMPLE 3:

Classify the nature of singularity of fn

$$f(z) = \frac{z - \sin z}{z^3}$$

Laurent series expansion about $z = 0$ is

$$f(z) = \frac{1}{z^3}\left\{ \cancel{z} - \cancel{z} + \frac{z^3}{3!} - \frac{z^5}{5!} + \frac{z^7}{7!} - \cdots \right\}$$

$$= \frac{1}{3!} - \frac{z^2}{5!} + \frac{z^4}{7!} \cdots$$

no (-)ve powers of $z \Rightarrow z = 0$ is removable singularity

THANK YOU !!