Definition

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Let $X, Y \subseteq \mathbb{R}$ and

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X\}$$

$$\tilde{B} = \{(y, \mu_{\tilde{B}}(y)) \mid y \in Y\}$$

Then $\tilde{R} = \{(x, y, \mu_{\tilde{R}}(x, y))|(x, y) \in X \times Y\}$ is a fuzzy relation on $\tilde{A}$ and $\tilde{B}$ if

$$\mu_{\tilde{R}}(x, y) \leq \mu_{\tilde{A}}(x), \ \forall (x, y) \in X \times Y$$

si

$$\mu_{\tilde{R}}(x, y) \leq \mu_{\tilde{B}}(y), \ \forall (x, y) \in X \times Y$$

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This definition is useful for fuzzy graphs, but we will not study fuzzy graphs in this course.

Union and intersection of fuzzy relations

Being fuzzy sets, between fuzzy relations defined on the same universes of discourse we can perform union and intersection operations as follows:

Definition

Let $\tilde{R}$ and $\tilde{Z}$ two fuzzy relations defined in the same product space. Union, respectively intersection of the fuzzy relations $\tilde{R}$ and $\tilde{Z}$ are defined:

$$\mu_{\tilde{R}\cup\tilde{Z}}(x,y) = \max\{\mu_{\tilde{R}}(x,y), \mu_{\tilde{Z}}(x,y)\}, \quad (x,y) \in X \times Y$$

$$\mu_{\tilde{R}\cap\tilde{Z}}(x,y) = \min\{\mu_{\tilde{R}}(x,y), \mu_{\tilde{Z}}(x,y)\}, \quad (x,y) \in X \times Y$$

Composition of fuzzy relations

- Discrete fuzzy relations, described by matrixes, can be composed in a similar way with matrixes multiplication
- Like in the case of matrixes, in order to compose the fuzzy relations, their dimensions have to match
- The most used composition method is the max − min composition
- Given two fuzzy relations expressed in matrix form, after the *max-min* composition it results a new fuzzy relation (a matrix) whose elements are obtained by "multiplication" of the elements of the two relations
- "Multiplication" is made between a line from the first matrix and a column from the second matrix, but instead of $+$ and $\cdot$ we use max and min
- There exists other composition methods than *max-min*, more precisely the *min* operation can be replaced by algebraic product, average, or other operations.
- Composition of fuzzy relations is important for understanding fuzzy inference.

The max-min composition

Definition

Given the fuzzy relations $\tilde{R}_1(x, y) \subset X \times Y$ and $\tilde{R}_2(y, z) \subset Y \times Z$, their max-min composition, $\tilde{R}_1$ *max-min* $\tilde{R}_2$, denoted $\tilde{R}_1 \circ \tilde{R}_2$ is defined as the fuzzy set:

$$\tilde{R}_1 \circ \tilde{R}_2 = \{((x, z), \mu_{\tilde{R}_1 \circ \tilde{R}_2}(x, z)) \mid (x, z) \in X \times Z\}$$

, where

$$\mu_{\tilde{R}_1 \circ \tilde{R}_2}(x, z) = \max_y [\min(\mu_{\tilde{R}_1}(x, y), \mu_{\tilde{R}_2}(y, z))]$$

The max-star composition

Definition

Given the fuzzy relations $\tilde{R}_1(x,y) \subset X \times Y$ and $\tilde{R}_2(y,z) \subset Y \times Z$, their max-star composition, $\tilde{R}_1 \circledast \tilde{R}_2$ is defined as the fuzzy set:

$$\tilde{R}_1 \circledast \tilde{R}_2 = \{((x,z), \mu_{\tilde{R}_1 \circledast \tilde{R}_2}(x,z)) \mid (x,z) \in X \times Z\}$$

, where

$$\mu_{\tilde{R}_1 \circledast \tilde{R}_2}(x,z) = \max_y [(\mu_{\tilde{R}_1}(x,y) * \mu_{\tilde{R}_2}(y,z))]$$

If the operation $*$ (*star*) is associative and monotonically nondecreasing in each argument, then the $\max - *$ composition has similar properties with the $\max - \min$ composition.
The most employed are the *max-prod* $(\max - \cdot)$(when the operation $*$ is the algebraic product, and *max-average*, when operation $*$ is the arithmetic mean.

Examples of compositions of discrete binary fuzzy relations

Let $\tilde{R}_1(x,y)$ and $\tilde{R}_2(y,z)$ discrete binary fuzzy relations defined by the following matrices:

$\tilde{R}_1$:

	y_1	y_2	y_3	y_4	y_5
x_1	0.1	0.2	0	1	0.7
x_2	0.3	0.5	0	0.2	1
x_3	0.8	0	1	0.4	0.3

$\tilde{R}_2$:

	z_1	z_2	z_3	z_4
y_1	0.9	0	0.3	0.4
y_2	0.2	0.1	0.8	0
y_3	0.8	0	0.7	1
y_4	0.4	0.2	0.3	0
x_5	0	1	0	0.8

Examples of compositions of discrete binary fuzzy relations

The results of max-min, max-prod and max-average composition of the relations $\tilde{R}_1(x, y)$ and $\tilde{R}_2(y, z)$ are given by the following matrices:

$\tilde{R}_1 \text{ max} - \text{min } \tilde{R}_2$

	z_1	z_2	z_3	z_4
x_1	0.4	0.7	0.3	0.7
x_2	0.3	1	0.5	0.8
x_3	0.8	0.3	0.7	1

$\tilde{R}_1 \text{ max} - \cdot\ \tilde{R}_2$

	z_1	z_2	z_3	z_4
x_1	0.4	0.7	0.3	0.56
x_2	0.27	1	0.4	0.8
x_3	0.8	0.3	0.7	1

Examples of compositions of discrete binary fuzzy relations

$\tilde{R}_1$ *max-average* $\tilde{R}_2$

	z_1	z_2	z_3	z_4
x_1	0.7	0.85	0.65	0.75
x_2	0.6	1	0.65	0.9
x_3	0.9	0.65	0.85	1

References

http://staff.cs.upt.ro/~todinca/cad/Lectures/cad_fuzzysets.pdf

https://www.tutorialspoint.com/fuzzy_logic/fuzzy_logic_quick_guide.htm

Thank you