Sylow p-subgroups:→

Let G be a group and p be a prime divisor of $o(G)$ such that $p^k | o(G)$ but $p^{k+1} \nmid o(G)$. Then a subgroup H of order p^k is called Sylow p-subgroups of G.

Example: Let $|G| = 2^2 \cdot 3$

Here $2^2 \mid |G|$ but $2^3 \nmid |G|$

and $3 \mid |G|$ but $3^2 \nmid |G|$.

Then there are Sylow 2-subgroup of order 4 and Sylow 3-subgroup of order 3.

Sylow's first theorem: (Partial Converse of Lagrange's theorem)

Let G be a finite group and p be a prime such that $p^m | o(G)$, for some +ve integer m, then there exist a subgroup H of G such that $o(H) = p^m$.

Proof: Let G be a finite group and p be a prime such that $p^m | o(G)$, for some +ve integer m.

Let $o(G) = n$.

We will prove the result by the method of induction on n.

If $n=1$, clearly the result holds.

If $n=2$, $2 | 2 = o(G)$. Also $G \leq G$

So the result holds in this case also.

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Suppose that the result holds for all groups having order $k < n$.

Let $p^m \mid o(G)$ and H be a subgroup of G s.t. $H \neq G$.

If $p^m \mid o(H) \leq n$, then by induction hypothesis, there exists a subgroup K of H order p^m.

Thus $K \leq G$ s.t. $o(K) = p^m$.

So the result holds in this case.

If $p^m \nmid o(H)$, $\forall\, H < G$. Then by Class equation,

$$|G| = |Z(G)| + \sum_{a \notin Z(G)} |G : C_G(a)|.$$

$$= |Z(G)| + \sum_{a \notin Z(G)} \frac{|G|}{|C_G(a)|}$$

Since $a \notin Z(G) \Rightarrow C_G(a) \neq G$

$\therefore\ C_G(a) < G$

$\therefore\ p^m \nmid C_G(a)$.

Now $|G| = \left|\frac{G}{C_G(a)}\right| \cdot |C_G(a)|$ and $p^m \mid |G|$.

$\Rightarrow p^m \mid \left|\frac{G}{C_G(a)}\right| < n$

$\Rightarrow p \mid \left|\frac{G}{C_G(a)}\right| \Rightarrow p \mid \sum_{a \notin C_G(a)} \frac{|G|}{|C_G(a)|}$

$\therefore\ p \mid |G| - \sum \left|\frac{G}{C_G(a)}\right| \Rightarrow p \mid |Z(G)|$

So by Cauchy's theorem, there exists an element $x \in Z(G)$ s.t. $o(x) = p$

Suppose $K = \langle x \rangle \subseteq Z(G)$

$\therefore\ |K| = p$.

Since any subgroup contained in the centre of a group is a normal subgroup of G, So $K \trianglelefteq G$.

Thus, we consider the quotient group $\frac{G}{K}$.

$$\left|\frac{G}{K}\right| < |G| = n$$

Also, $|G| = \left|\frac{G}{K}\right| . |K|$ and $p^m \mid |G|$.

$\Rightarrow p^m \mid \left|\frac{G}{K}\right| . |K|$ and $p^m \nmid |K|$

So $p^{m-1} \mid p^m \mid \left|\frac{G}{K}\right|$.

By Induction hypothesis, there is a subgroup $\frac{H}{K}$ of $\frac{G}{K}$, where $H \leq G$ such that $\left|\frac{H}{K}\right| = p^{m-1}$

Now $|H| = \left|\frac{H}{K}\right| . |K| = p^{m-1} . p = p^m$.

So G has a subgroup of order p^m.

Thus the result is true in this case.

Hence the result is true for all group.

<u>Remarks</u>: This theorem shows ~~that~~ the existence of p-subgroups of a group G. ie

If $o(G) = 120 = 2^3 \times 3 \times 5$.

Then there must be at least one subgroup of order 2, 4, 8, 3 and 5.

But it doesnot say the existence of subgroups of order 6, 10, 15.

In the continuation, we say that there is a Sylow 2-subgroup of order 8, Sylow 3-subgroup of order 3 and Sylow 5-subgroup of order 5.

Sylow's Second theorem
Let G be a finite group and H be any p-subgroups of G, Then H is contained in some Sylow p-subgroup.

Sylow's Third theorem:
Let G be a finite group, the number of Sylow p-subgroup is $1 \pmod{p}$ ie. of the form $1+kp$, for some $k \in \mathbb{N}$ and $(1+kp) \mid o(G)$.
Moreover, any two Sylow p-subgroups are conjugate to each other.

For proof of these two theorem, refer to J. Gallian.

lemma: The number of Sylow p-subgroups of G is equal to $\frac{o(G)}{o(N(P))}$, where P is a Sylow p-subgroup of G.

Proof: Let G be a finite group and P be a Sylow p-subgroup of G
Since $|cl(P)| = \frac{|G|}{|N(P)|}$

And $cl(P) = \{Q \leq G : Q = gPg^{-1}, g \in G\}$
= Set of all Sylow p-subgroups as any two Sylow p-subgroups are conjugate to each other.

So Number of Sylow p-subgroup $= \frac{|G|}{|N(P)|}$.

* If P is the only Sylow p-subgroup of G then $\left|\frac{G}{N(P)}\right| = 1 \Rightarrow |G| = |N(P)|$
$\Rightarrow G = N(P)$ $(\because N(P) \leq G)$
$\Rightarrow$ P is normal in G.

Conversely, if P is normal in G, then
$G = N(P)$
$\Rightarrow \left|\frac{G}{N(P)}\right| = 1$
$\Rightarrow$ P is the only Sylow p-subgroup.

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<u>Theorem:</u> Let $o(G) = pq$, where p and q are distinct primes such that $p < q$ and $p \nmid (q-1)$. Then G is cyclic.

<u>Proof:</u> Let $o(G) = pq$, where p and q are distinct prime s.t. $p<q$ and $p \nmid (q-1)$.

The number of Sylow p-subgroup of $G = 1+kp$ & $(1+kp) \mid q$

$\Rightarrow 1+kp = q$ or $1+kp = 1$

$\Rightarrow p \mid q-1$ or $k = 0$

$\therefore k = 0$ $[\because p \nmid (q-1)]$

So, the number of Sylow p-subgroup of $G = 1$

Thus Sylow p-subgroup of G is normal.

Now, the number of Sylow q-subgroup of $G = 1+k'q$ and $(1+k'q) \mid p$

$\Rightarrow 1+k'q = p$ or $1+k'q = 1$

$\Rightarrow q \mid p-1$ or $k' = 0$

$\Rightarrow k' = 0$ $[\because p<q]$

So, the number of Sylow q-subgroup of $G = 1$

Thus Sylow q-subgroup of G is normal in G.

Let H be a Sylow p-subgroup and K be a Sylow q-subgroup of G. So $|H| = p$ and $|K| = q$. Hence H and K are both cyclic subgroup. $H = \langle a \rangle$ and $K = \langle b \rangle$

Also $H \trianglelefteq G$ and $K \trianglelefteq G$.

So $HK \trianglelefteq G$ $\Rightarrow hk = kh$ for all $h \in H, k \in K$.

$$|HK| = \frac{|H||K|}{|H \cap K|} = |H||K| \quad [\text{as } |H \cap K| = 1]$$

$\Rightarrow |HK| = pq = |G|$

$\therefore HK = G$

Now $|a| = |H| = p$ and $|b| = |K| = q$. So $ab = ba$

So ~~[illegible]~~ $\gcd(|a|, |b|) = \gcd(p, q) = 1$.

So $|ab| = \text{lcm}(|a|, |b|) = pq = |G|$. So G is cyclic.

Result: Let G be a finite group, then $o\left(\frac{G}{Z(G)}\right) \neq pq$, for any two distinct primes p & q with $p<q$ & $p \nmid (q-1)$.

Sol^n: If possible, let $o\left(\frac{G}{Z(G)}\right) = pq$, $p<q$ & $p \nmid (q-1)$.

By previous result, $\frac{G}{Z(G)}$ is cyclic.

$\Rightarrow$ G is abelian

$\Rightarrow$ $G = Z(G)$

$\Rightarrow o\left(\frac{G}{Z(G)}\right) = 1$, a contradiction.

Thus $o\left(\frac{G}{Z(G)}\right) \neq pq$, for distinct primes p & q with $p<q$ & $p \nmid (q-1)$.

Problem: Let $o(G)=30$. Show that

(i) Either Sylow 3-subgroup or Sylow 5-subgroup is normal in G

(ii) G has a normal subgroup of order 15.

(iii) Both Sylow 3-subgroup and Sylow 5-subgroup are normal in G.

Solution: Given $o(G) = 30 = 2\times 3\times 5$

(i) The number of Sylow 3-subgroups $= 1+3k$ and $(1+3k) \mid 10$

$\Rightarrow k = 0$ or 3.

If $k=0$, the Sylow 3-subgroup is normal.

If $k \neq 0$ then $k=3$, then number of Sylow 3-subgroups of order 3 $= 10$,

So there are 20 distinct elements of order 3.

The number of Sylow 5-subgroups $= 1+5k'$ and $(1+5k') \mid 6$

$\Rightarrow k' = 0, 1$.

If $k'=0$, the Sylow 5-subgroup is normal.

If $k' \neq 0$ then $k'=1$, the number of Sylow 5-subgroups of order 5 $= 6$

So there are 24 distinct elements of order 5.

Thus, we have more than 44 $(20+24)$ elements in G, a contradiction

Hence, either Sylow 3-subgroup or Sylow 5-subgroup is normal in G.

(ii) Let H be a Sylow 3-subgroup of order 3 and K be a Sylow 5-subgroup of order 5. Also one of them is normal in G. So HK is a subgroup of G.

Now $o(H\cap K) \mid o(H)=3$ and $o(H\cap K) \mid o(K)=5$

$\Rightarrow o(H\cap K) = 1$

$\therefore o(HK) = \dfrac{o(H)\,o(K)}{o(H\cap K)} = \dfrac{3\times 5}{1} = 15$

Index of HK in G is 2. So HK is normal in G.

(iii) Suppose H is normal in G and K is not normal in G.

So G has 24 elements of order 5.

Also $o(HK) = 15 = 3\times 5$ and $3 \nmid (5-1)$.

So HK is cyclic subgroup of order 15. So it has $\phi(15) = 8$ elements of order 15.

Thus G has more than $24+8 = 32$ elements, a contradiction.

So both H and K are normal in G.

* Hence a group of order 30 is not 'Simple'.

Problem: Let G be a group of order 231. Show that 11 Sylow subgroup of G is contained in the centre of G.

Solⁿ: Given $o(G) = 231 = 3\times 7\times 11$

The number of Syllow 11-subgroups of $G = 1+11k$ and $(1+11k) \mid 21$

$\Rightarrow k = 0$.

So Syllow 11-subgroup H of G is normal and $o(H) = 11$

The number of Sylow 7-subgroups of $G = 1+7k'$ and $(1+7k') \mid 33$

$\Rightarrow k' = 0$

So Sylow 7-subgroup K is normal in G and $o(K) = 7$.

$\therefore o\left(\frac{G}{K}\right) = 33 = 3\times 11$ and $3 \nmid (11-1)$. So $\frac{G}{K}$ is cyclic.

So $\frac{G}{K}$ is abelian.

We know that commutator subgroup G' is the smallest subgroup of G such that $\frac{G}{G'}$ is abelian.

So $G' \subseteq K \Rightarrow o(G') \mid o(K) \Rightarrow o(G') = 1$ or 7.

If $o(G') = 1$, then $G' = \{e\} \Rightarrow x^{-1}y^{-1}xy = e \Rightarrow xy = yx, \forall x, y \in G$

$\Rightarrow G$ is abelian

$\Rightarrow G = Z(G)$.

$\therefore H \subseteq Z(G)$.

If $o(G') = 7 \Rightarrow G' = K$.

Clearly $H \cap K = \{e\}$ $[\because o(H) = 11, o(K) = 7]$

Let $x \in H, y \in G$. Then $x^{-1}y^{-1}xy \in G' = K$

$\Rightarrow x^{-1}(y^{-1}xy) \in H$ $[\because H$ is normal in $G]$

$\Rightarrow x^{-1}y^{-1}xy \in H \cap K = \{e\}$

$\Rightarrow xy = yx$, for all $x \in H, y \in G$.

$\therefore H \subseteq Z(G)$,

Assignment

Discuss the simplicity of a group of order 36, 56, 108, 144.