Solution of Spherical, Cylindrical wave equation with Eigen value and function

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Spherical wave equation

The wave eqn

$$u_{tt} - c^2(u_{xx} + u_{yy} + u_{zz}) = 0 \quad \text{---} (1)$$

In spherocal polar coordinates (r, θ, ϕ),

$$x = r\cos\theta \sin\phi$$
$$y = r\sin\theta \sin\phi$$
$$z = r\cos\phi$$

the wave equation (1) takes the form

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}$$
$$= \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \quad \text{———} (2)$$

Solutions of the equation (2) are called spherical symmetric waves if u depends on r and t only.

Therefore, the solution $u = u(r,t)$ which satisfies the wave equation with spherical symmetry in three-dimensional space is with spherical symmetry in three-dimensional space is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \quad \text{--- (3)}$$

Let $U = r\,u(r,t)$ be the new dependent variable, the

$$\frac{\partial^2 U}{\partial t^2}=c^2\frac{\partial^2 U}{\partial r^2} \quad \text{--- (4)}$$

We know that the one-dimensional (1-D) wave eqn $\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}$

and the general solution of the form

$$u(x,t)=\phi(x+ct)+\psi(x-ct)$$

Then, the general solution of the form eqn (4) reduce to

$$U(r,t) = \phi(r+ct) + \psi(r-ct) \quad \text{--- (5)}$$

or

$$u(r,t) = \frac{1}{r}\left[\phi(r+ct) + \psi(r-ct)\right] \quad \text{--- (6)}$$

This solution contains only two progressive spherical waves traveling with constant velocity c. The terms involving ϕ and ψ represent the incoming waves to the origin and the outgoing waves from the origin respectively.

Therefore, the solution for only outgoing waves generated incoming waves to the origin and the outgoing waves from the origin respectively.

Therefore, the solution for only outgoing waves generated by a source

$$u(r,t) = \frac{1}{r}\psi(r-ct) \quad (7)$$

where ψ is to be determined from the properties of the source. In the case of fluid flows, u shows the velocity potential so that the limiting total flux through a sphere of ~~centre~~ center at the origin

$$Q(t) = \lim_{r\to 0} 4\pi r^2 u_r(r,t)$$

$$= \lim_{r\to 0} 4\pi r^2 \times \left\{-\frac{1}{r^2}\psi(r-ct) + \frac{1}{r}\psi'\right\}$$

$$Q(t) = -4\pi\,\psi(-ct) \quad (8)$$

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Note that:-

The Source Strength Q, is the multiple 4π (product) of the surface area and normal surface velocity of the monopole.

The solution of the IVP with initial condition

$$u(r,0) = f(r), \quad u_t(r,0) = g(r), \; r \geq 0 \quad \text{— (9)}$$

Now from eqn (6) and (9), we have

$$u(r,0) = f(r) = \frac{1}{r}\left[\phi(r) + \psi(r)\right]$$

$$\phi(r) + \psi(r) = r f(r) \quad \text{— (10)}$$

$$u_t(r,t) = \frac{c}{r}\left[\phi'(r+ct) - \psi'(r-ct)\right]$$

$$u_t(r,0) = \frac{c}{r}\left[\phi'(r) - \psi'(r)\right] = g(r) \quad \text{— (11)}$$

Integrating eqn (11), we have

$$u_t(x,0) = \frac{c}{x}\left[\phi'(x) - \psi'(x)\right] = g(x) \quad \text{---} (11)$$

Integrating eqn (11), we have

$$\phi(x) - \psi(x) = \frac{1}{c}\int_{x_0}^{x} \alpha g(\alpha)\, d\alpha + k \quad \text{---} (12)$$

Solving eqn (10) and (12), ~~xxxx~~

we get

$$2\phi(x) = x f(x) + \frac{1}{c}\int_{x_0}^{x} \alpha g(\alpha)\, d\alpha + k$$

$$\phi(x) = \frac{1}{2}\left[x f(x) + \frac{1}{c}\int_{x_0}^{x} \alpha g(\alpha)\, d\alpha + k\right]$$

Similarly

$$\psi(x) = \frac{1}{2}\left[x f(x) - \frac{1}{c}\int_{x_0}^{x} \alpha g(\alpha)\, d\alpha - k\right]$$

Therefore,

$$u(x,t) = \frac{1}{2x}\left[(x+ct) f(x+ct) + (x-ct) f(x-ct) + \frac{1}{c}\left\{\int_{x_0}^{x+ct} \alpha g(\alpha)\, d\alpha - \int_{x_0}^{x-ct} \alpha g(\alpha)\, d\alpha\right\}\right]$$

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$$u(r,t) = \frac{1}{2r}\left[(r+ct)f(r+ct) + (r-ct)f(r-ct) + \frac{1}{c}\int_{r-ct}^{r+ct} \alpha g(\alpha)\,d\alpha\right] \quad (13)$$

$r \geq ct$.

Therefore, when $r < ct$, then this Solution fails because f and g are not defined for $r < 0$.

Thus initial value at $t=0$, $r \geq 0$ determine the Solution $u(r,t)$ only up to the characterist ie. $r = ct$ in the $r-t$ plane.

The Solution for $U(r,t)$

$$U(r,t) = \frac{1}{2}\left[(r+ct)f(r+ct) + (r-ct)f(r-ct) + \frac{1}{c}\int_{r-ct}^{r+ct} \alpha g(\alpha)\,d\alpha\right]$$

$$U(r,t) = \frac{1}{2}\left[(r+ct)f(r+ct) + (r-ct)\right.$$
$$\left. f(r-ct) + \frac{1}{c}\int_{r-ct}^{r+ct} \alpha g(\alpha)\,d\alpha\right] \quad \text{--- (14)}$$

when $r \geq ct \geq 0$ and

$$U(r,t) = \frac{1}{2}\left[\phi(ct+r) + \psi(ct-r)\right],$$
when $ct \geq r \geq 0$ --- (15)

where $\phi(ct) + \psi(ct) = 0$, for $ct \geq 0$

--- (16)

Similarly

we can write eqn (15), then

$$u(r,t) = \frac{1}{2r}\left[(ct+r)f(ct+r) - (ct-r)\right.$$
$$\left. f(ct-r) + \frac{1}{c}\int_{ct-r}^{ct+r} \alpha g(\alpha)\,d\alpha\right]$$

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Cylindrical wave equation:-

let $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$, be cylindrical polar coordinates (r, θ, z), the wave equation $(u_{tt} = c^2 u_{xx})$

Now assume the form

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz} = \frac{1}{c^2}u_{tt}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}u_{\theta\theta} + u_{zz} = \frac{1}{c^2}u_{tt} \quad \text{--- (1)}$$

if u depends only on r and t, then the above equation (1) becomes

$$u_{rr} + \frac{1}{r}u_r = \frac{1}{c^2}u_{tt} \quad \text{--- (2)}$$

The solutions of eqn (2) are called cylindrical waves becomes

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The solutions of eqn (2) are called cylindrical waves

For a periodic solution in time

$$u(r,t) = F(r) e^{i\omega t} \quad \text{—— (3)}$$

Then $\dfrac{\partial u}{\partial r} = F'(r) e^{i\omega t}$

$$\frac{\partial^2 u}{\partial t^2} = -\omega^2 F(r) e^{i\omega t}$$

putting the above values, eqn (2)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r F'(r) e^{i\omega t}\right] = -\frac{\omega^2}{c^2} F(r) e^{i\omega t}$$

$$\frac{1}{r}\left[1 \cdot F'(r) e^{i\omega t} + F''(r)\, r e^{i\omega t}\right] +$$

$$+ \frac{\omega^2}{c^2} F(r) e^{i\omega t} = 0$$

$$F''(r) + \frac{F'(r)}{r} + \frac{\omega^2}{c^2} F(r) = 0 \quad \text{--- (4)}$$

which has the form of Bessel's equation.

The solution of eqn (4) is

$$F(r) = A J_0\left(\frac{\omega r}{c}\right) + B Y_0\left(\frac{\omega r}{c}\right) \quad \text{--- (5)}$$

$$F(r) = A_1\left[J_0\left(\frac{\omega r}{c}\right) + i Y_0\left(\frac{\omega r}{c}\right)\right] + B_1\left[J_0\left(\frac{\omega r}{c}\right) - i Y_0\left(\frac{\omega r}{c}\right)\right] \quad \text{--- (6)}$$

(complex form)

The above eqn (6) can be rewritten as

$$F(r) = A_1 H_0^{(1)}\left(\frac{\omega r}{c}\right) + B_1 H_0^{(2)}\left(\frac{\omega r}{c}\right) \quad \text{--- (7)}$$

rewritten as

$$F(r) = A_1 H_0^{(1)}\left(\frac{\omega r}{c}\right) + B_1 H_0^{(2)}\left(\frac{\omega r}{c}\right) \quad \text{--- (7)}$$

where

$$H_0^{(1)} = J_0\left(\frac{\omega r}{c}\right) + iY_0\left(\frac{\omega r}{c}\right) \quad \text{--- (8)}$$

$$H_0^{(2)} = J_0\left(\frac{\omega r}{c}\right) \pm iY_0\left(\frac{\omega r}{c}\right), \quad \text{--- (9)}$$

$H_0^{(1)}, H_0^{(2)}$ are Hankel functions defined by abeve expression eq^n (8) & (9).

The Solution of 1-D wave eq^n

$$u(r,t) = \left[A_1 H_0^{(1)}\left(\frac{\omega r}{c}\right) + B_1 H_0^{(2)}\left(\frac{\omega r}{c}\right)\right] e^{i\omega t}$$

we knew that asymptotic expression for $H_0^{(1)}$ and $H_0^{(2)}$

(7) $H_0^{(1)}(\theta) = \sqrt{\frac{2}{\pi\theta}}\, e^{i(\theta - \frac{\pi}{4})}$

$H_0^{(2)}(\theta) = \sqrt{\frac{2}{\pi\theta}}\, e^{-i(\theta - \frac{\pi}{4})}$

The general periodic solution to the given wave eqn in cylindrical coordinates

$$u(r,t) = \sqrt{\frac{2c}{\pi\omega}} \left[A_1 e^{-\frac{i\pi}{4}} \frac{e^{i(\frac{\omega}{c})(r+ct)}}{\sqrt{r}} + B_1 e^{\frac{i\pi}{4}} e^{-i(\frac{\omega}{c})(r-ct)} \right]$$

Sturm–Liouville theory

Consider the second-order differential equation of the

Sturm–Liouville theory

Consider the second-order differential equation of the form

$$a_1(x)\frac{d^2y}{dx^2}+a_2(x)\frac{dy}{dx}+[a_3(x)+\lambda]y=0 \quad (1)$$

if $p(x)=e^{\int^x \frac{a_2(t)}{a_1(t)}dt}$,

$$q(x)=\frac{a_3(x)}{a_1(x)}p(x),\quad s(x)=\frac{p(x)}{a_1(x)} \quad (2)$$

we obtain

$$\frac{d}{dx}\left(p\frac{dy}{dx}\right)+(q+\lambda s)y=0 \quad (3)$$

which is known as the Sturm–Liouville equation

where

$$L\equiv\frac{d}{dx}\left(p\frac{d}{dx}\right)+q$$

the above eqn can be written as

$$L[y] + \lambda s(x) y = 0 \quad \text{--- (4)}$$

where λ is a parameter independent of x, and p, q and s are real valued function of x. The Sturm-Liouville equation is called regular in the interval $[a,b]$ if the functions $p(x)$ and $s(x)$ are positive in the interval $[a,b]$.

For a given λ, there exist two L.I. solutions of a regular Sturm-Liouville regular Sturm-Liouville equation in the interval $[a, b]$.

Now, eqn (4) rewriting

$$L[y] + \lambda s(x) y = 0, \quad a \le x \le b \quad \text{--- (5)}$$

with

$$\left.\begin{aligned} a_1 y(a) + a_2 y'(a) &= 0, \\ b_1 y(b) + b_2 y'(b) &= 0 \end{aligned}\right] \text{--- (6)}$$

where a_1 and a_2 and likewise b_1 and b_2 are not both zero and are given real numbers, is called a regular Sturm-Liouville system.

The values of λ for which the Sturm-Liouville system has a non-trivial solution

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~~Note~~ (9)

are called the eigenvalues and the corresponding solutions are called the eigenfunctions.

~~Ex: Sol Solve~~

~~Sol~~

Solve the regular Sturm-Liouville problem

(1) $y'' + \lambda y = 0, \quad 0 \le x \le \pi$

$y(0) = 0, \quad y'(\pi) = 0$

(2) $x^2 y'' + x y' + \lambda y = 0$

or

$\frac{d}{dx}\left(x \frac{dy}{dx}\right) + \frac{1}{x} \lambda y = 0 \qquad 1 \le x \le e$

$y(1) = 0, \quad y(e) = 0$

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(3) $y'' + \lambda y = 0, \; -\pi \le x \le \pi$

$y(-\pi) = y(\pi), \; y'(-\pi) = y'(\pi)$

Let $\phi(x)$ and $\psi(x)$ be any real-valued integrable functions on an interval I. Then ϕ and ψ are said to be orthogonal on I with respect to a weight function $\rho(x) > 0$, iff

$$\langle \phi, \psi \rangle = \int_I \phi(x)\, \psi(x)\, \rho(x)\, dx = 0 \quad \text{(7)}$$

The interval I may be of infinite or it may be either open or closed at one or both ends of the finite interval when $\phi = \psi$ in eqn (7),

then the norm of ϕ

$$\|\phi\| = \left[\int_I \phi^2(x)\, s(x)\, dx \right]^{1/2} \quad \text{--- (8)}$$

Th:- Let the Coefficients p, q and s in the Sturm-Liouville system be continuous in $[a, b]$. Let the eigenfunctions ϕ_j and ϕ_k, corresponding to λ_j and λ_k be continuously differentiable. Then ϕ_j and ϕ_k are orthogonal with respect to the weight function $s(x)$ in $[a, b]$.

eigenfunctions ϕ_j and ϕ_k, corresponding to λ_j and λ_k be continuously differentiable. Then ϕ_j and ϕ_k are orthogonal with respect to the weight function $s(x)$ in $[a, b]$.

Th: All the eigenvalues of a regular Sturm-Liouville System with $s(x) > 0$ are real.

Th: If $\phi_1(x)$ and $\phi_2(x)$ are any two solutions of the equation $L[y] + \lambda s(x) y = 0$ on $[a, b]$, then $p(x) W(\phi_1, \phi_2)(x) =$ constant, where W is the Wronskian.

Th: An eigenfunction of a regular Sturm-Liouville is unique except for a constant factor.