Solution of VIE of the Second kind by Successive approximation and problems

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Solution of Volterra Integral Equation of the Second kind Successive approximations

Consider Volterra integral equation of the second kind

$$y(x) = f(x) + \lambda \int_a^x K(x,t)\, y(t)\, dt \quad \text{—(1)}$$

let $y_0(x) = f(x)$ —(2)

if $y_n(x)$ and $y_{n-1}(x)$ are the n^{th} order and $(n-1)^{th}$ order approximations respectively,

$$y_n(x) = f(x) + \lambda \int_a^x K(x,t)\, y_{n-1}(t)\, dt \quad \text{—(3)}$$

n^{th} order and $(n-1)^{th}$ order approximations respectively,

$$y_n(x) = f(x) + \lambda \int_a^x K(x,t)\, y_{n-1}(t)\, dt \quad \text{--- (3)}$$

We know that the iterated kernels or iterated functions $K_n(x,t)$, $(n = 1, 2, 3, \ldots)$ are defined $K_1(x,t) = K(x,t)$ — (4)

and $$K_n(x,t) = \int_t^x K(x,z)\, K_{n-1}(z,t)\, dz \quad \text{--- (5)}$$

putting $n=1$ in eqn (3), the first order approximation $y_1(x)$ is given by

$$y_1(x) = f(x) + \lambda \int_a^x K(x,t)\, y_0(t)\, dt \quad \text{--- (6)}$$

From eqn (2), $y_0(t) = f(t)$ — (7)

Substituting the above value of

2

value of $y_0(t)$ in eqn (6), we get

$$y_1(x) = f(x) + \lambda \int_a^x K(x,t) f(t)\, dt \quad \text{--- (8)}$$

putting $n=2$ in eqn (3), the second-order approximation $y_2(x)$ is given by $y_2(x) = f(x) + \lambda \int_a^x K(x,t) y_1(t)\, dt$

$$y_2(x) = f(x) + \lambda \int_a^x K(x,z) y_1(z)\, dz \quad \text{--- (9)}$$

Replacing x by z in eqn (8), we have

$$y_1(z) = f(z) + \lambda \int_a^z K(z,t) f(t)\, dt \quad \text{--- (10)}$$

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Replacing x by z in eqn (8),

we have

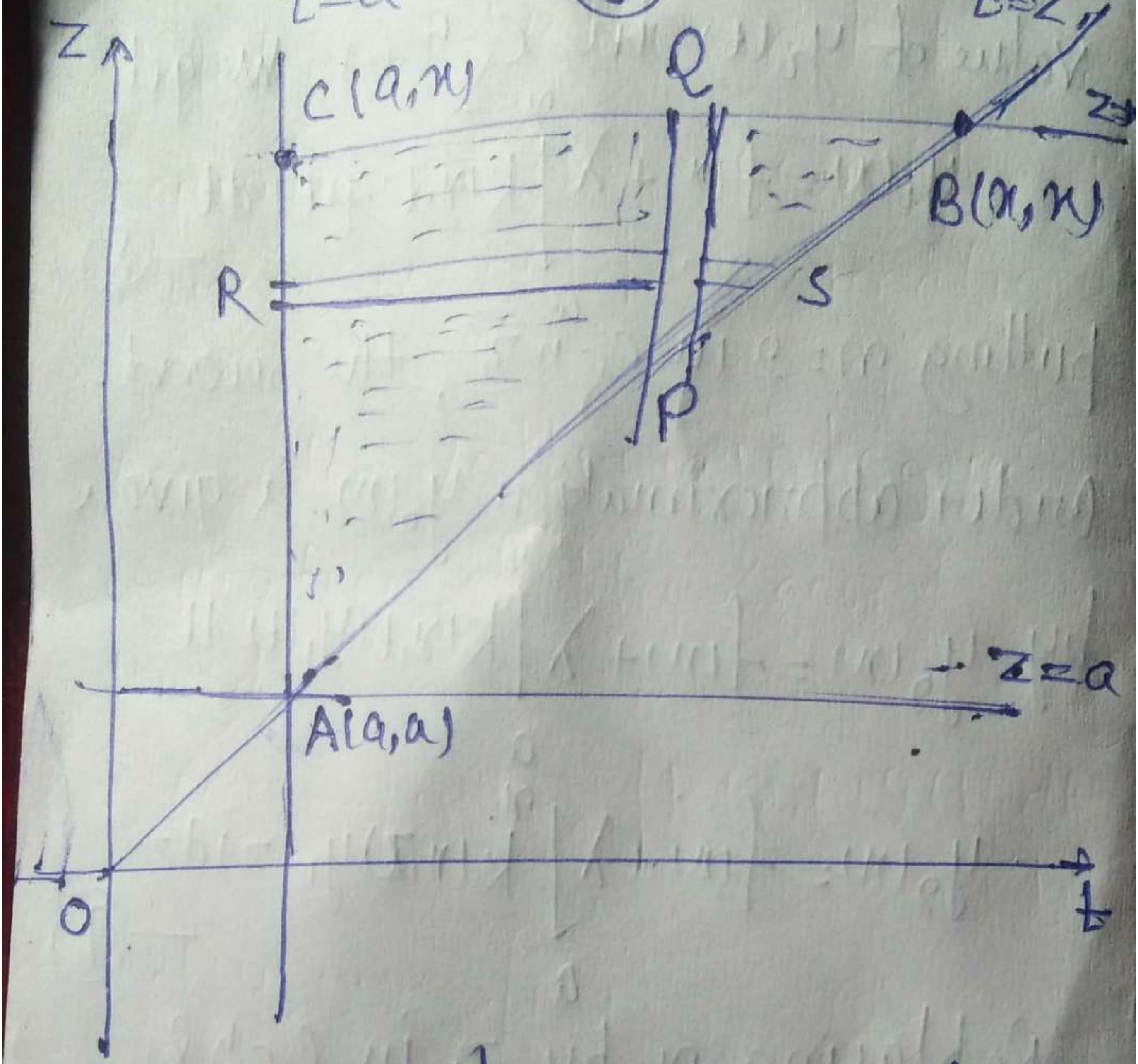
$$y_1(z) = f(z) + \lambda \int_a^z k(z,t)\, f(t)\, dt \qquad \text{— (10)}$$

Substituting the above value of $y_1(z)$ in eqn (9), we obtain

$$y_2(x) = f(x) + \lambda \int_a^x k(x,z)\left[f(z) + \lambda \int_a^z k(z,t)\, f(t)\, dt\right] dz$$

$$y_2(x) = f(x) + \lambda \int_a^x k(x,z)\, f(z)\, dz + \lambda^2 \int_{z=a}^{x} k(x,z)\left[\int_{t=a}^{z} k(z,t)\, f(t)\, dt\right] dz \qquad \text{— (11)}$$

Now, consider the double integral on the RHS of eqn (11).



In double integral, clearly strip RS || t-axis

and strip PQ || z-axis

Then the limits of z are z = t to z = x and limits for t are t = a to t = x.

and Strip PQ || z-axis
Then the limits of z are $z = t$ to $z = x$ and limits for t are $t = a$ to $t = x$.

Therefore,

$$\int_{z=a}^{x} k(x,z)\left[\int_{t=a}^{z} k(z,t) f(t)\, dt\right] dz$$

$$= \int_{t=a}^{x} f(t)\left[\int_{z=t}^{x} k(x,z) k(z,t)\, dz\right] dt \quad \text{--- (12)}$$

using the above expression eqn (12) in eqn (11), we obtain

$$y_2(x) = f(x) + \lambda \int^{x} k(x,z) f(z)\, dz +$$

$$+ \lambda^2 \int_{t=a}^{x} f(t)\left[\int_{z=t}^{x} k(x,z) k(z,t)\, dz\right] dt$$

$$y_2(x) = f(x) + \lambda \int_a^x K(x,t) f(t)\, dt + \lambda^2 \int_a^x f(t) K_2(x,t)\, dt$$ (4)

[For $n = 2$, in eqn (5), then

$$K_2(x,t) = \int_t^x K(x,z) K_1(z,t)\, dz$$

using eqn (4)

$$K_2(x,t) = \int_t^x K(x,z) K(z,t)\, dz$$]

$$y_2(x) = f(x) + \sum_{m=1}^{2} \lambda^m \int_a^x K_m(x,t) f(t)\, dt \quad \text{--- (13)}$$

Proceeding same as above,

$$y_n(x) = f(x) + \sum_{m=1}^{n} \lambda^m \int_a^x K_m(x,t) f(t)\, dt$$

Proceeding same as above,

$$y_n(x) = f(x) + \sum_{m=1}^{n} \lambda^m \int_a^x K_m(x,t)\, f(t)\, dt \quad -(14)$$

taking the limit as $n \to \infty$, we obtain

$$y(x) = \lim_{n \to \infty} y_n(x)$$

$$= f(x) + \sum_{m=1}^{\infty} \lambda^m \int_a^x K_m(x,t)\, f(t)\, dt \quad -(15)$$

To determine the resolvent kernel $R(x,t,\lambda)$ in term of the iterated kernels $K_m(x,t)$.

$$y(x) = f(x) + \lambda \int_a^x \left[\sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t) \right] f(t)\, dt$$

$$y(x) = f(x) + \lambda \int_a^x R(x,t;\lambda)\, f(t)\, dt$$

5

Q Solve the integral equation

$$y(x) = 1+x^2 + \int_0^x \frac{1+x^2}{1+t^2} y(t)\,dt \quad -(1)$$

Solⁿ: Given that eqⁿ (1).

We know that standard form Integral eqⁿ

$$y(x) = f(x) + \lambda \int_0^x K(x,t)\, y(t)\,dt$$

where $f(x) = 1+x^2$, $\lambda = 1$,
$K(x,t) = (1+x^2)/(1+t^2)$] - (2)

Let $K_m(x,t)$ be the m^{th} iterated Kernel, then $K_1(x,t) = K(x,t)$

and

$$K_m(x,t) = \int_t^x K(x,z)\, K_{m-1}(z,t)\,dz \quad -(3)$$

kernel, then $K_1(x,t) = K(x,t)$

and $K_m(x,t) = \int_t^x K(x,z) K_{m-1}(z,t)\, dz$ — (3)

$K_1(x,t) = K(x,t) = (1+x^2)/(1+t^2)$ — (3*)

Putting $m=2$ in eqn (3), we have

$$K_2(x,t) = \int_t^x K(x,z) K_1(z,t)\, dz$$

$$= \int_t^x \left(\frac{1+x^2}{1+z^2}\right)\cdot\left(\frac{1+z^2}{1+t^2}\right) dz = \int_t^x \frac{1+x^2}{1+t^2}\, dz$$

$$K_2(x,t) = \frac{1+x^2}{1+t^2}(x-t) \quad \text{— (4)}$$

Next, putting $m=3$ in eqn (3), we obtain

$$K_3(x,t) = \int_t^x K(x,z) K_2(z,t)\, dz$$

$$= \int_t^x \left(\frac{1+x^2}{1+z^2}\right)\left(\frac{1+z^2}{1+t^2}\right)(z-t)\, dz$$

(6)

$$K_3(x,t) = \frac{1+x^2}{1+t^2}\int_t^x (z-t)\,dz$$

$$K_3(x,t) = \frac{1+x^2}{1+t^2}\left[\frac{(z-t)^2}{2}\right]_t^x = \frac{1+x^2}{1+t^2}\cdot\frac{(x-t)^2}{2!} \quad \text{—(5)}$$

Similarly, $m=4$,

$$K_4(x,t) = \frac{1+x^2}{1+t^2}\,\frac{(x-t)^3}{3!} \quad \text{—— (6)}$$

and so on.

$$K_m(x,t) = \frac{1+x^2}{1+t^2}\cdot\frac{(x-t)^{m-1}}{(m-1)!},\ m=1,2,3\ldots \quad \text{—— (7)}$$

The resolvent kernel $R(x,t;\lambda)$,

we have

$$R(x,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t),\ (\because \lambda=1)$$

$$= \sum_{m=1}^{\infty} K_m(x,t)$$

$$= K_1(x,t) + K_2(x,t) + K_3(x,t) + \ldots$$

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$m=1$

$$= \sum_{m=1}^{\infty} K_m(x,t)$$

$$= K_1(x,t) + K_2(x,t) + K_3(x,t) + \cdots$$

$$= \frac{1+x^2}{1+t^2} + \frac{1+x^2}{1+t^2}\frac{(x-t)}{1!} + \frac{1+x^2}{1+t^2}\frac{(x-t)^2}{2!} + \cdots$$

$$= \frac{1+x^2}{1+t^2}\left[1 + \frac{(x-t)}{1!} + \frac{(x-t)^2}{2!} + \frac{(x-t)^3}{3!} + \cdots\right]$$

$$R(x,t;\lambda) = \frac{1+x^2}{1+t^2} e^{(x-t)}.$$

The required solution of eq^n (1)

$$y(x) = f(x) + \lambda \int_0^x R(x,t;\lambda) f(t)\,dt$$

$$y(x) = 1+x^2 + 1.\int_0^x \frac{1+x^2}{1+t^2}\cdot e^{x-t}\cdot(1+t^2)\,dt$$

$$y(x) = 1+x^2 + (1+x^2)e^x \int_0^x e^{-t}\,dt$$

$$\boxed{y(x) = e^x(1+x^2)}$$

Ans

7

Q Solve $y(x) = x + \int_0^x (t-x)\,y(t)\,dt$ —(1)

Soln: Given that equation (1)

Comparing eqn (1) with

$y(x) = f(x) + \lambda \int_0^x K(x,t)\,y(t)\,dt$ —(2)

$f(x) = x$, $\lambda = 1$, $K(x,t) = t - x$

let $K_m(x,t)$ be the mth iterated kernel, then $K_1(x,t) = K(x,t)$ —(3)

and $K_m(x,t) = \int_t^x K(x,z)\,K_{m-1}(z,t)\,dz$, —(4)

$m = 2, 3, 4, \ldots$

Now from eqn (2) and (3), we get

$K_1(x,t) = K(x,t) = t - x$ —(5)

putting $m=2$ in eqn (4) and using eqn (5), we obtain

Now from eqn (2) and (3), we get

$$K_1(x,t) = K(x,t) = t-x \quad \text{--- (5)}$$

putting $m=2$ in eqn (4) and using eqn (5), we obtain

$$K_2(x,t) = \int_t^x K(x,z)K_1(z,t)\,dz$$

$$K_2(x,t) = \int_t^x (z-x)(t-z)\,dz$$

$$K_2(x,t) = (t-z)\frac{(z-x)^2}{2}\Big|_t^x - \int_t^x (-1)\frac{(z-x)^2}{2}\,dz$$

$$K_2(x,t) = \frac{(t-x)^3}{3!} \quad \text{--- (6)}$$

putting $m=3$ in eqn (4), we have

$$K_3(x,t) = \int_t^x K(x,z)K_2(z,t)\,dz = \frac{1}{3!}\int_t^x (z-x)(t-z)^3\,dz$$

$$K_3(x,t) = \frac{(t-x)^5}{5!} \quad \text{--- (7)}$$

- - - - - - and so on.

$$K_m(x,t) = (-1)^{m-1} \frac{(t-x)^{2m-1}}{(2m-1)!}, \quad m = 1,2,3 \ldots$$

The resolvent kernel

$$R(x,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t)$$

$$= K_1(x,t) + K_2(x,t) + K_3(x,t) + \ldots$$

$$= \frac{(t-x)}{1!} - \frac{(t-x)^3}{3!} + \frac{(t-x)^5}{5!} - \ldots$$

$$R(x,t;\lambda) = \sin(t-x).$$

The required solution of

eqn ① $y(x) = f(x) + \lambda \int_0^x R(x,t;\lambda) f(t)\,dt$

$$y(x) = x + \int_0^x \sin(t-x) \cdot t\,dt$$

eqⁿ ① $y(x) = f(x) + \lambda \int_0^x R(x,t;\lambda) f(t) dt$

$y(x) = x + \int_0^x \sin(t-x) \cdot t \, dt$

$y = y(x) = x + t\left\{-\cos(t-x)\right\}_0^x$

$- \int_0^x 1 \left\{-\cos(t-x)\right\} dt$

$$\boxed{y(x) = \sin x}$$

Q① Solve $y(x) = 1 + \int_0^x (t-x) y(t) dt$ — ①

$\boxed{\text{Ans } y = \cos x}$

Solⁿ ~~Given the integral eqn~~ ①

Q② Solve

$y(x) = \cos x - x - 2 + \int_0^x (t-x) y(t) dt$

Ans

$$\boxed{y(x) = -\cos x - \sin x - \frac{x}{2} \sin x}$$

9

Q Find the Neumann Series for the Solution of the integral equation

$$y(x) = 1 + x + \lambda \int_0^x (x-t)\, y(t)\, dt \quad -(1)$$

Solⁿ: Given that equation (1), we know that Volterra integral equation

$$y(x) = f(x) + \lambda_1 \int_a^x K(x,t)\, y(t)\, dt \quad (2)$$

Now from equation (1) and (2), we get $f(x) = 1+x$, $\lambda_1 = \lambda$ and $K(x,t) = x - t$ -(3*) } -(3)

Let $K_m(x,t)$ be the mth iterated kernel, then

$K_1(x,t) = K(x,t)$ and (4)

x

and $K(x,t) = x - t$ — (3*), ∫ — (3)

Let $K_m(x,t)$ be the m-th iterated kernel, then

$$K_1(x,t) = K(x,t) \text{ and}$$ — (4)

$$K_m(x,t) = \int_t^x K(x,z) K_{m-1}(z,t)\, dz$$ — (5)

From eqn (3*) and (4),

$$K_1(x,t) = K(x,t) = x - t$$ — (6)

putting $m = 2$ in eqn (5) and using eqn (6), we have

$$K_2(x,t) = \int_t^x K(x,z) K_1(z,t)\, dz$$

$$K_2(x,t) = \int_t^x (x-z)(z-t)\, dz$$

$$K_2(x,t) = \frac{(x-t)^3}{3!} \quad \text{--- (7)}$$

putting $m=3$ in eqⁿ (5),

we have

$$K_3(x,t) = \int_t^x K(x,z)\, K_2(z,t)\, dz$$

$$K_3(x,t) = \frac{(x-t)^5}{5!} \quad \text{--- (8)}$$

Now, the Neumann Series for the solution of eqⁿ (1)

$$y(x) = f(x) + \sum_{m=1}^{\infty} \lambda^m \int_0^x K_m(x,t)\, f(t)\, dt$$

$$y(x) = 1+x+\lambda \int_0^x K_1(x,t)(1+t)\, dt + \lambda^2 \int_0^x K_2(x,t)(1+t)\, dt + \lambda^3 \int_0^x K_3(x,t)(1+t)\, dt$$

$+ \ldots$

$$y(x) = 1+x+\lambda\int_0^x K_1(x,t)(1+t)\,dt + \lambda^2\int_0^x K_2(x,t)(1+t)\,dt + \lambda^3\int_0^x K_3(x,t)(1+t)\,dt + \cdots$$

$$y(x) = 1+x+\lambda\int_0^x (x-t)(1+t)\,dt + \lambda^2\int_0^x \frac{(x-t)^3}{3!}(1+t)\,dt + \lambda^3\int_0^x \frac{(x-t)^5}{5!}(1+t)\,dt + \cdots$$

$$y(x) = 1+x+\lambda\left(\frac{x^2}{2}+\frac{x^3}{6}\right) + \frac{\lambda^2}{3!}\left[\frac{x^4}{4}+\frac{x^5}{20}\right] + \frac{\lambda^3}{5!}\left(\frac{x^6}{6}+\frac{x^7}{42}\right) + \cdots$$

$$y(x) = 1+x+\lambda\left(\frac{x^2}{2!}+\frac{x^3}{3!}\right) + \lambda^2\left(\frac{x^4}{4!}+\frac{x^5}{5!}\right) + \lambda^3\left(\frac{x^6}{6!}+\frac{x^7}{7!}\right) + \cdots$$

— (9)

If $\lambda = 1$, then eqn (9) reduces to

$$y(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = e^x$$

~~A~~ by resolvent kernel

$$R(x,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t)$$

$$R(x,t;\lambda) = (x-t) + \lambda \frac{(x-t)^3}{3!} + \lambda^2 \frac{(x-t)^5}{5!} + \cdots$$

whose sum cannot be obtained in closed form.

Therefore, the solution cannot be obtained by the usual formula

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obtained in closed form.
Therefore, the solution cannot be obtained by the usual formula.

Q Solve the Volterra integral equation

$$y(x) = 1 + \int_0^x xt\, y(t)\, dt \quad \text{—(b)}$$

Ans:-

$$y(x) = 1 + \frac{x^3}{2} + \frac{x^6}{2.5} + \frac{x^9}{2.5.8} + \frac{x^{12}}{2.5.8.11} + \cdots$$

(12)

Q. Using the method of successive approximations, solve the integral equation

① $y(x) = 1 + \int_0^x y(t)\,dt, \quad y_0(x) = 0$

Ans $y(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x$

② $y(x) = 1 + \int_0^x (x-t)\,y(t)\,dt,$

$y_0(x) = 1.$

Ans $y(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2n-2}}{(2n-2)!} + \cdots$

$\cdots +$ ad inf.

or

$y(x) = \cosh x$

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$y(x) = \cosh x$

(3) $y(x) = x - \int_0^x (x-t)\, y(t)\, dt,$

$y_0(x) = 0.$

Ans

$y(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$+ (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} + \cdots$ ad inf

$y(x) = \sin x$

(4) $y(x) = 1 + x - \int_0^x y(t)\, dt, \; y_0(x) = 1.$

Ans $y(x) = 1$