Random Variables and probability Distributions-III

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Moment Generating function: Let f(x,y) denote the joint probability density function of the two random variables X and Y. If $E[e^{t_1 X + t_2 Y}]$ exists for $-h_1 < t_1 < h_1$, $-h_2 < t_2 < h_2$, where h_1 and h_2 are positive, it is denoted by $M(t_1, t_2)$ and is called the moment-generating function of the joint distribution of X and Y.

Hence, the Marginal Distribution of X and Y are

$$M(t_1, 0) = E[e^{t_1 X}] = M(t_1)$$

and $$M(0, t_2) = E[e^{t_2 Y}] = M(t_2)$$

Also,

$$E(X^K Y^m) = \left[\frac{\partial^{k+m}}{\partial t_1^k \partial t_2^m} M(t_1, t_2) \right]_{t_1 = 0, t_2 = 0}$$

Independence of Two Random Variables:

(1) Two random variables X and Y, forming a **discrete random variable**, are independent if and only if: $p_{ij} = p_{*j} \cdot p_{i*}$

where p_{ij} is their joint probability mass function and p_{i*} and p_{*j} are their marginal probability mass functions.

(2) Two random variables X and Y, forming an absolutely **continuous random variable**, are independent if and only if: $f(x, y) = f_X(x) f_Y(y)$

where $f(x, y)$ is their joint probability mass function and $f_X(x)$ and $f_Y(y)$ are their marginal probability mass functions.

- Let the stochastically independent random variables X and Y have the marginal probability density functions $f_X(x)$ and $f_Y(y)$, respectively. Then $E[XY] = E[X]E[Y]$

Example: Let

$$f(x,y)=\begin{cases} e^{-y}, \; 0<x<y<\infty \\ 0, \;\; elsewhere \end{cases}$$

be the joint probability density function of X and Y. Find the moment generating function of this distribution.

Solution: Moment Generating Function of X and Y,

$$M(t_1,t_2)=E[e^{t_1X+t_2Y}]$$

$$=\int_0^{\infty}\int_0^{y} e^{t_1x+t_2y} f(x,y)dxdy$$

$$=\int_0^{\infty}\int_0^{y} e^{t_1x+t_2y}e^{-y}dxdy$$

$$=\int_0^{\infty}\int_0^{y} e^{t_1x+(t_2-1)y}dxdy$$

$$=\int_0^{\infty} e^{(t_2-1)y}\left[\frac{e^{t_1x}}{t_1}\right]_0^{y}dy$$

$$=\int_0^{\infty} e^{(t_2-1)y}\left[\frac{e^{t_1y}-1}{t_1}\right]_0^{y}dy$$

$$M(t_1, t_2) = \frac{1}{t_1} \int_0^\infty \left[e^{-(1-t_1-t_2)y} - e^{-(1-t_2)y} \right] dy$$

$$= \frac{1}{t_1} \left[\frac{1}{1-t_1-t_2} - \frac{1}{1-t_2} \right]$$

$$= \frac{1}{(1-t_1-t_2)(1-t_2)}$$

Discrete Probability Distributions

1-Bernoulli Distribution: A random variable X is said to have a Bernoulli distribution with parameter p if its probability mass function is given by:

$$P(X = x) = \frac{p^x (1-p)^{1-x}}{x!}$$

, for $x = 0,1$

$=0$ otherwise.

The parameter p satisfies $0 \le p \le 1$. Often $(1-p)$ is denoted as q.

2-Binomial Distribution: If X is discrete random variable which can take values 0,1,2,3,…,n such that $P(X = x) = nC_x p^x q^{n-x}$, x = 0,1,2,….,n where $p + q = 1$ then X is said to follow a Binomial distribution with parameters n and p.

Moment Generating Function of Binomial Distribution:

$$M(t) = E[e^{tX}]$$

$$= \sum_{x=0}^{n} e^{tx} p_x$$

$$= \sum_{x=0}^{n} e^{tx} nC_x p^x q^{n-x}$$

$$= \sum_{x=0}^{n} nC_x (pe^t)^x q^{n-x}, \qquad \because (a+b)^n = \sum_{r=0}^{n} nC_r a^r b^{n-r}$$

$$= (pe^t + q)$$

Derivatives of MGF,

$$M'(t) = n(pe^t + q)^{n-1} \times pe^t$$

$$M''(t) = np[(pe^t + q)^{n-1} \times e^t + (n-1)(pe^t + q)^{n-2} pe^{2t}]$$

Mean and Variance of Binomial Distribution

$$E(X) = M'(0) = np$$

$$E(X^2) = M''(0) = np[1 + (n-1)p]$$

$$Var(X) = E(X^2) - [E(X)]^2 = npq$$

Mean : np

$Variance : npq$

Example: In a large consignment of electric bulbs 10 % are defective. A random sample of 20 is taken for inspection. Find the probability that (i) All are good bulbs (ii) at most 3 are defective bulbs (iii) Exactly there are three defective bulbs.

Solution: Let X be the event of defective bulbs,

$$p = 0.1 \quad q = 0.9 \quad n = 20$$

(i) $P(X = 0) = 20C_0\,(0.1)^0\,(0.9)^{20} = 0.1216$

(ii) $P(X \leq 3) = p(0) + p(1) + p(2) + p(3)$

$$= 20C_0\,(0.1)^0\,(0.9)^{20} + 20C_1\,(0.1)^1\,(0.9)^{19}$$

$$+ 20C_2\,(0.1)^3\,(0.9)^{18} + 20C_3\,(0.1)^3\,(0.9)^{17}$$

$$= 0.8666$$

(iii) $P(X = 3) = 20C_3\,(0.1)^3\,(0.9)^{17} = 0.19.$

3-Negative Binomial Distribution: Let $p(x)$ be the probability that exactly $x + r$ trails will be required to produce r success. Clearly the last trial must be a success and the probability is p. In the remaining $x + r - 1$ trials, there must be $r - 1$ successes and the probability of this is given by

$$p(X = x) = \begin{cases} (x + r - 1)C_{r-1} p^r q^x, & x = 0,1,2,3,\ldots \\ 0, & otherwise \end{cases}$$

Moment Generating Function of Negative Binomial Distribution:

$$M(t) = \sum_{x=0}^{\infty} e^{tx} p_x$$

$$= \sum_{x=0}^{\infty} e^{tx} (x + r - 1)C_{r-1} p^r q^x$$

$$= p^r \sum_{x=0}^{\infty} (x + r - 1)C_{r-1} \left(qe^t\right)^x$$

$$= p^r \begin{bmatrix} (r-1)C_0 (qe^t)^0 + (r)C_1 (qe^t)^1 + (r+1)C_2 (qe^t)^2 \\ + (r+2)C_3 (qe^t)^3 + \ldots. \end{bmatrix}$$

$$= p^r \left[1 + \frac{r}{1!}(qe^t) + \frac{(r+1)r}{2!}(qe^t)^2 + \frac{(r+2)(r+1)r}{3!}(qe^t)^3 + \ldots. \right]$$

$$\because (1-a)^{-n} = 1 + \frac{n}{1!}a + \frac{n(n+1)}{2!}a^2 + \frac{n(n+1)(n+2)}{3!}a^3 + ...$$

$$M(t) = p^r(1-qe^t)^{-r} \quad \text{, for} \quad t < -\log_e q$$

Derivatives of MGF,

$$M'(t) = p^r(-r)(1-qe^t)^{-r-1}(-qe^t) = rp^r qe^t(1-qe^t)^{-r-1}$$

$$M''(t) = rp^r q\left[e^t(-r-1)(1-qe^t)^{-r-2}(-qe^t) + e^t(1-qe^t)^{-r-1}\right]$$

$$= rp^r q(1-qe^t)^{-r-1}e^t[(r+1)(1-qe^t)^{-1}(qe^t)+1]$$

Mean and Variance of Negative Binomial Distribution:

Mean, $E[X] = M'(0) = rp^r qe^0 (1 - qe^0)^{-r-1}$

$$= rp^r q(1-q)^{-r-1}$$

$$= rp^r qp^{-r-1}$$

$$= \frac{rq}{p}$$

$$E[X^2] = M''(0) = rp^r q(1-qe^0)^{-r-1} e^0 [(r+1)(1-qe^0)^{-1}(qe^0) + 1]$$

$$= rp^r qp^{-r-1}[(r+1)p^{-1}q + 1]$$

$$= rqp^{-1}[(r+1)p^{-1}q + 1]$$

$$= \frac{r(r+1)q^2}{p^2} + \frac{rq}{p}$$

Variance ,

$$Var[X] = E[X^2] - E[X]^2$$

$$= \frac{r(r+1)q^2}{p^2} + \frac{rq}{p} - \frac{r^2q^2}{p^2}$$

$$= \frac{rq^2}{p^2} + \frac{rq}{p}$$

$$= rq\left(\frac{q+p}{p^2}\right)$$

$$= \frac{rq}{p^2}$$

Example: Find the probability that in tossing 4 coins one will get either all heads or all tails for the third time on the seventh toss.

Solution: P(H H H H) = 1/16;   P(T T T T) = 1/16

P(all head $\cup$ all tail) = 1/16 + 1/16 = 1/8

$\therefore\ p = 1/8,\ q = 7/8$; $x + r = 7,\ r = 3$

$$P(X = x) = (x + r - 1)C_{r-1} p^r q^x$$

$$P(X = 4) = 7 - 1C_{3-1}\left(\frac{1}{8}\right)^3\left(\frac{7}{8}\right)^4$$

$$= 6C_2\left(\frac{1}{8}\right)^3\left(\frac{7}{8}\right)^4$$

$$= 0.0169$$

Example: In a company 5% defective components are produced. What is the probability that atleast 5 components are to be examined in order to get three defectives?

Solution:

Given, $p = 0.05$, $q = 0.95$; $x + r \leq 5$, $r = 3$

$$P(X = x) = (x + r - 1)C_{r-1} p^r q^x$$

$P(X \geq 2) = 1 - P(X < 2)$

$= 1 - P(X = 0) - P(X = 1)$

$= 1 - 2C_2 (0.05)^3 (0.95)^0 - 3C_2 (0.05)^3 (0.95)^1$

$= 0.9995.$

4-Trinomial Distribution: The binomial distribution can be generalized to the trinomial distribution. The random variables X and Y is said to have trinomial distribution is if they have the joint probability density function f(x,y) given by,

$$f(x,y)=\frac{n!}{x!\,y!\,(n-x-y)!}\,p_1^x\,p_2^y\,p_3^{n-x-y}$$

where *x* and *y* are non-negative integers with

$x+y\le n$ and p_1, p_2, p_3 are positive proper fraction with $p_1+p_2+p_3=1$

5-Multinomial Distribution: The trinomial distribution is generalized to the multinomial distribution as follows:

If a given trial can result in the k outcomes $E_1, E_2, \ldots, E_k$ with probabilities $p_1, p_2, \ldots, p_k$, then the probability distribution of the random variables $X_1, X_2, \ldots, X_k$, representing the number of occurrences for $E_1, E_2, \ldots, E_k$ in n independent trials, is

$$f(x_1, x_2, \ldots, x_k) = \frac{n!}{x_1! x_2! \ldots x_{k-1}! x_k!} p_1^{x_1} p_2^{x_2} \ldots p_{k-1}^{x_{k-1}} p_k^{x_k}$$

with $\sum_{i=0}^{k-1} x_i \leq n$ and $\sum_{i=0}^{k} p_i = 1$

Moment Generating Function of Multinomial Distribution:

$$M(t_1, t_2, \ldots t_{k-1}) = \left[p_1 e^{t_1} + p_2 e^{t_2} + \ldots + p_{k-1} e^{t_{k-1}} + p_k\right]^n$$

6-Poisson Distribution: If X is a discrete random variable that can assume the values 0,1,2,… such that its probability mass function is given by

$$P(X = x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \quad , x = 0,1,2,\ldots; \lambda > 0.$$

Then X is said to follow a Poisson distribution with parameter λ .

Poisson distribution is a limiting case of binomial distribution under the following assumptions.

- The number of trials 'n' should be indefinitely large. i.e., $n \to \infty$.
- The probability of successes 'p' for each trial is indefinitely small.
- $np = \lambda$, should be finite where λ is a constant.

Moment Generating Function of Poisson Distribution:

$$M(t) = \sum_{x=0}^{\infty} e^{tx} p_x$$

$$= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(e^t \lambda)^x}{x!}, \qquad \because e^a = \sum_{n=0}^{\infty} \frac{a^n}{n!}$$

$$= e^{-\lambda} e^{e^t \lambda}$$

$$= e^{\lambda(e^t - 1)}$$

Mean and Variance of Poisson Distribution:

$$M'(t) = e^{-\lambda} e^{e^t \lambda} \lambda e^t$$

$$M''(t) = \lambda e^{-\lambda} \left[e^t (e^{e^t \lambda} \lambda e^t) + e^{e^t \lambda} e^t \right]$$

$$E[X] = M'(0) = e^{-\lambda} e^{e^0 \lambda} \lambda e^0 \lambda = \lambda$$

$$E[X^2] = M''(0) = \lambda e^{-\lambda} \left[e^0 (e^{e^0 \lambda} \lambda e^0) + e^{e^0 \lambda} e^0 \right] = \lambda e^{-\lambda} \left[\lambda e^{\lambda} + e^{\lambda} \right] = \lambda^2 + \lambda$$

Mean , $E[X] = \lambda$

Variance, $Var[X] = E[X^2] - E[X]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$

Example: A random variable X follows Poisson distribution and if P(X=1) = 2P(X=2),

find (i) P(X = 0)

(ii) S.D. of X.

Solution: Given that P(X=1) = 2P(X=2)

$$\frac{e^{-\lambda}\lambda^{1}}{1!} = 2\frac{e^{-\lambda}\lambda^{2}}{2!}$$

$$\Rightarrow \lambda = 1$$

(i) $P(X = 0) = \dfrac{e^{-\lambda}\lambda^{0}}{0!} = e^{-1} = 0.3679$

(ii) S.D. of X $= \sqrt{\text{var}\, X} = \sqrt{\lambda} = 1$

Continuous Probability Distribution:

1-Exponential distribution: A random variable X is said to have exponential distribution with parameter $\alpha > 0$ if its probability density function is given by

$$f(x) = \alpha\, e^{-\alpha\, x}, x \geq 0$$

$$= 0, \quad \text{otherwise}$$

Gamma Function: In Integral Calculus, the integral

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y}\, dy, \qquad \alpha > 0$$

is called the gamma function, with

1. $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$
2. $\Gamma(\alpha+1) = \alpha!$
3. $\Gamma(1) = 1$
4. $\Gamma(1/2) = \sqrt{\pi}$

2- Gamma Distribution: A random variable X is said to have gamma distribution with parameter α if its probability density function is given by

$$f(x) = \begin{cases} \dfrac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\Gamma(\alpha)\beta^{\alpha}}, & 0 < x < \infty \\ 0, & elsewhere \end{cases}$$

Moment Generating function of Gamma Distribution:

$$M(t) = E[e^{tX}]$$

$$= \int_0^\infty e^{tx} \frac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\Gamma(\alpha)\beta^\alpha} dx$$

$$= \int_0^\infty \frac{x^{\alpha-1} e^{-x\left(\frac{1-\beta t}{\beta}\right)}}{\Gamma(\alpha)\beta^\alpha} dx$$

$$\int_0^\infty \frac{\left(\frac{\beta y}{1-\beta t}\right)^{\alpha-1} e^{-y}}{\Gamma(\alpha)\beta^\alpha} \left(\frac{\beta y}{1-\beta t}\right) dy$$

$$Let\ \ y = \frac{1-\beta t}{\beta} x$$

$$\Rightarrow x = \frac{\beta}{1-\beta t} y$$

$$\therefore dx = \frac{\beta}{1-\beta t} dy$$

$$M(t) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}}\left(\frac{\beta}{1-\beta t}\right)^{\alpha}\int_0^{\infty} y^{\alpha-1}e^{-y}dy$$

$$= \frac{1}{\Gamma(\alpha)\beta^{\alpha}}\left(\frac{\beta}{1-\beta t}\right)^{\varepsilon}\Gamma(\alpha)$$

$$= (1-\beta t)^{-\alpha}, \qquad if\ t < \frac{1}{\beta}$$

Also $\quad M'(t) = \alpha\beta(1-2\beta t)^{-\alpha-1}$

$$M''(t) = \alpha(\alpha+1)\beta^2(1-\beta t)^{-\alpha-2}$$

Mean, $\quad E(X) = M'(0) = \alpha\beta$

$$E(X^2) = M''(0) = \alpha(\alpha+1)\beta^2$$

Variance, $\quad Var(X) = E(X^2) - E(X)^2 = \alpha^2\beta^2 + \alpha\beta^2 - \alpha^2\beta^2 = \alpha\beta^2$

3-Normal Distribution: Let X be a continuous random variable have a normal distribution with parameter a (mean) and b^2 (variance) if its probability density function is given by the probability law:

$$f(x) = \frac{1}{b\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-a}{b}\right)^2} \ , \ -\infty < x < \infty, b > 0$$

Moment Generating Function of Normal Distribution: The moment generating function is

$$M(t) = e^{at+\frac{b^2t^2}{2}}$$

and

$$M'(t) = (a + b^2 t)e^{at+\frac{b^2t^2}{2}}$$

$$M''(t) = (a + b^2 t)^2 e^{at+\frac{b^2t^2}{2}} + b^2 e^{at+\frac{b^2t^2}{2}}$$

Mean, $\mu = E[X] = M'(0) = a$

$$E[X^2] = M''(0) = a^2 + b^2$$

Variance, $\sigma^2 = Var[X] = E[X^2] - E[X]^2 = a^2 + b^2 - a^2 = b^2$

Theorem: If the random variable X is n(μ,σ^2) then the random variable W=(X - μ)/σ is n(0,1)

Proof: Let G(w) and g(w) be the distribution and density function of W and W=(X - μ)/σ.

$$G(w) = P[W \le w]$$

$$= P\left[\frac{X-\mu}{\sigma} \le w\right]$$

$$= P[X \le \mu + \sigma w]$$

$$= \int_{x=-\infty}^{x=\mu+\sigma w} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Let, $y = \frac{x-\mu}{\sigma}$ $i.e, x = \mu + y\sigma, \therefore dx = \sigma dy$

when $x = -\infty, y = -\infty; x = \mu + w\sigma, y = w$

Hence

$$G(w) = \int_{y=-\infty}^{y=w} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}y^2} \sigma\, dy$$

$$g(w) = G'(w) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}w^2}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{w-0}{1}\right)^2},\ -\infty < w < \infty$$

which is n(0,1)

Bivariate Normal Distribution: Let X and Y be two random variables having the joint probability density function

$$f(x,y)=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}e^{-\frac{q}{2}},\quad -\infty<x<\infty,\ -\infty<y<\infty$$

where $q=\dfrac{1}{1-\rho^2}\left[\left(\dfrac{x-\mu_1}{\sigma_1}\right)^2-2\rho\left(\dfrac{x-\mu_1}{\sigma_1}\right)\left(\dfrac{y-\mu_2}{\sigma_2}\right)+\left(\dfrac{y-\mu_2}{\sigma_2}\right)^2\right],$

$$\sigma_1>0,\ \sigma_2>0,\ -1<\rho<1$$

then X and Y are said to have a bivariate normal distribution.

Theorem: Let X and Y have a bivariate normal distribution. Prove that marginal probability density function of X and Y are respectively

$n(\mu_1, \sigma_1^2)$ and $n(\mu_2, \sigma_2^2)$ and ρ is the correlation coefficient between X and Y.

Proof: Marginal density function of X,

$$f_1(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{q}{2}} dy$$

where

$$q = \frac{1}{1-\rho^2}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]$$

$$(1-\rho^2)q = \left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2$$

$$= \left[\left(\frac{y-\mu_2}{\sigma_2}\right) - \rho\left(\frac{x-\mu_1}{\sigma_1}\right)\right]^2 - \rho^2\left(\frac{x-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x-\mu_1}{\sigma_1}\right)^2$$

$$= \frac{1}{\sigma_2}\left[y-\mu_2-\rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)\right]^2 + (1-\rho^2)\left(\frac{x-\mu_1}{\sigma_1}\right)^2$$

$$(1-\rho^2)q = \frac{1}{\sigma_2}\left[y-b\right]^2 + (1-\rho^2)\left(\frac{x-\mu_1}{\sigma_1}\right)^2, \quad b = \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)$$

$$\therefore \frac{q}{2} = \frac{1}{2(1-\rho^2)}\left[\frac{y-b}{\sigma_2}\right]^2 + \frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2$$

thus $$f_1(x) = \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{q}{2}} dy$$

$$= \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2}\left(\frac{y-b}{\sigma_2\sqrt{1-\rho^2}}\right)^2} dy$$

$$= \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)}, \qquad \because \int_{-\infty}^{\infty} f(x)dx = 1,$$

Normal p.d.f. with mean b and variance $\sigma_2^2(1-\rho^2)$

Now,

$$f(y/x) = \frac{f(x,y)}{f_1(x)} = \frac{1}{\sqrt{2\pi}\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2}\left(\frac{y-b}{\sigma_2\sqrt{1-\rho^2}}\right)^2}, -\infty < y < \infty$$

Which is $n(b, \sigma_2^2(1-\rho^2))$

Here *b* is the conditional mean of Y given X = x,

$$b = E[Y/x] = \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)$$

Similarly,

$$E[X/y] = \mu_1 + \rho\frac{\sigma_1}{\sigma_2}(y-\mu_2)$$

Coefficient of x in E[Y/x] × Coefficient of y in E[X/y]

$$= \rho\frac{\sigma_2}{\sigma_1} \times \rho\frac{\sigma_1}{\sigma_2} = \rho^2$$

Moment Generating Function of Bivariate Normal Distribution:

$$M(t_1, t_2) = E[e^{t_1 X + t_2 Y}]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1 x + t_2 y} f(x, y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty} e^{t_1 x} f_1(x)\left[\int_{-\infty}^{\infty} e^{t_2 y} f(y/x)\,dx\right]dy, \quad \because f(y/x) = \frac{f(x, y)}{f_1(x)}$$

Since $\int_{-\infty}^{\infty} e^{t_2 y} f(y/x)\,dx$ is the moment generating function of the conditional probability function f(y/x). Also f(y/x) is a normal p.d.f. with mean $\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x - \mu_1)$ and variance $\sigma_2^2(1 - \rho^2)$.

$$\int_{-\infty}^{\infty} e^{t_2 y} f(y/x)dx = e^{\left\{t_2\left[\mu_2+\rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)\right]+\frac{t^2\sigma_2^2}{2}(1-\rho^2)\right\}}$$

thus,

$$M(t_1,t_2) = \int_{-\infty}^{\infty} e^{t_1 x} f_1(x) e^{\left\{t_2\left[\mu_2+\rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)\right]+\frac{t_2^2\sigma_2^2}{2}(1-\rho^2)\right\}} dx$$

$$= e^{\left\{\left[\mu_2 t_2-\rho\frac{\sigma_2}{\sigma_1}\mu t_2\right]+\frac{t_2^2\sigma_2^2}{2}(1-\rho^2)\right\}} \int_{-\infty}^{\infty} e^{\left(t_1+t_2\rho\frac{\sigma_2}{\sigma_1}\right)x} f_1(x)\,dx$$

Also $f_1(x)$ is the normal p.d.f. with mean μ_1 variance σ_1^2

$$\int_{-\infty}^{\infty} e^{\left(t_1+t_2\rho\frac{\sigma_2}{\sigma_1}\right)x} f_1(x)\,dx = e^{\left(\mu_1\left(t_1+t_2\rho\frac{\sigma_2}{\sigma_1}\right)+\frac{\sigma_1^2}{2}\left(t_1+t_2\rho\frac{\sigma_2}{\sigma_1}\right)^2\right)}$$

$$M(t_1,t_2) = e^{\left\{\left[\mu_2 t_2 - \rho\frac{\sigma_2}{\sigma_1}\mu t_2)\right] + \frac{t_2^2\sigma_2^2}{2}(1-\rho^2)\right\}} e^{\left(\mu_1\left(t_1 + t_2\rho\frac{\sigma_2}{\sigma_1}\right) + \frac{\sigma_1^2}{2}\left(t_1 + t_2\rho\frac{\sigma_2}{\sigma_1}\right)^2\right)}$$

$$= e^{\left(\mu_1 t_1 + \mu_2 t_2 + \frac{\sigma_1^2 t_1^2 + \sigma_2^2 t_2^2 + 2\rho\sigma_1\sigma_2 t_1 t_2}{2}\right)}$$

Which is the moment generating function of bivariate normal distribution. It is to note that, if $\rho = 0$, then $M(t_1,t_2) = M(t_1,0)M(0,t_2)$.

Thus X and Y are independent when $\rho = 0$.

Transformation of Random Variables: If X and Y are random variables with joint probability density function $f_{xy}(x,y)$ and if $Z = g(X,Y)$ and $W = h(X,Y)$ are two other random variables, then the joint probability density function of Z and W is given by $f_{zw}(z,w) = f_{xy}(x,y)\left|J\right|$

where

$$J = \frac{\partial(x,y)}{\partial(z,w)} = \begin{vmatrix} \dfrac{\partial x}{\partial z} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial z} & \dfrac{\partial y}{\partial w} \end{vmatrix}$$

is called the Jacobian of the transformation

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THANK YOU