Subject: Financial Economics
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Topic: Mean-variance portfolio theory
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Expected portfolio return with *ex ante* probabilities

- Investing usually needs to deal with uncertain outcomes.
- That is, the unrealized return is usually random, and can take on any one of a finite number of specific values, say $r_1, r_2, \ldots, r_S$.
- This randomness can be described in probabilistic terms. That is, for each of these possible outcomes, they are associated with a probability, say $p_1, p_2, \ldots, p_S$.
- $p_1 + p_2 + \ldots, + p_S = 1$.
- For asset i, its expected return is: $E(r_i) = p_1 * r_1 + p_2 * r_2 + \ldots + p_S * r_S$.
- The expected rate of return a portfolio of n assets, $E(r) = w_1 * E(r_1) + w_2 * E(r_2) + \ldots + w_n * E(r_n)$.

Variability measures with *ex ante* probabilities

- $\text{Var}(r) = p_1 * (\text{E}(r) - r_1)^2 + p_2 * (\text{E}(r) - r_2)^2 + \ldots + p_S * (\text{E}(r) - r_S)^2$.
- $\text{Std}(r) = \text{Var}(r)^{1/2}$.
- These formulas apply to both individual assets and portfolios.

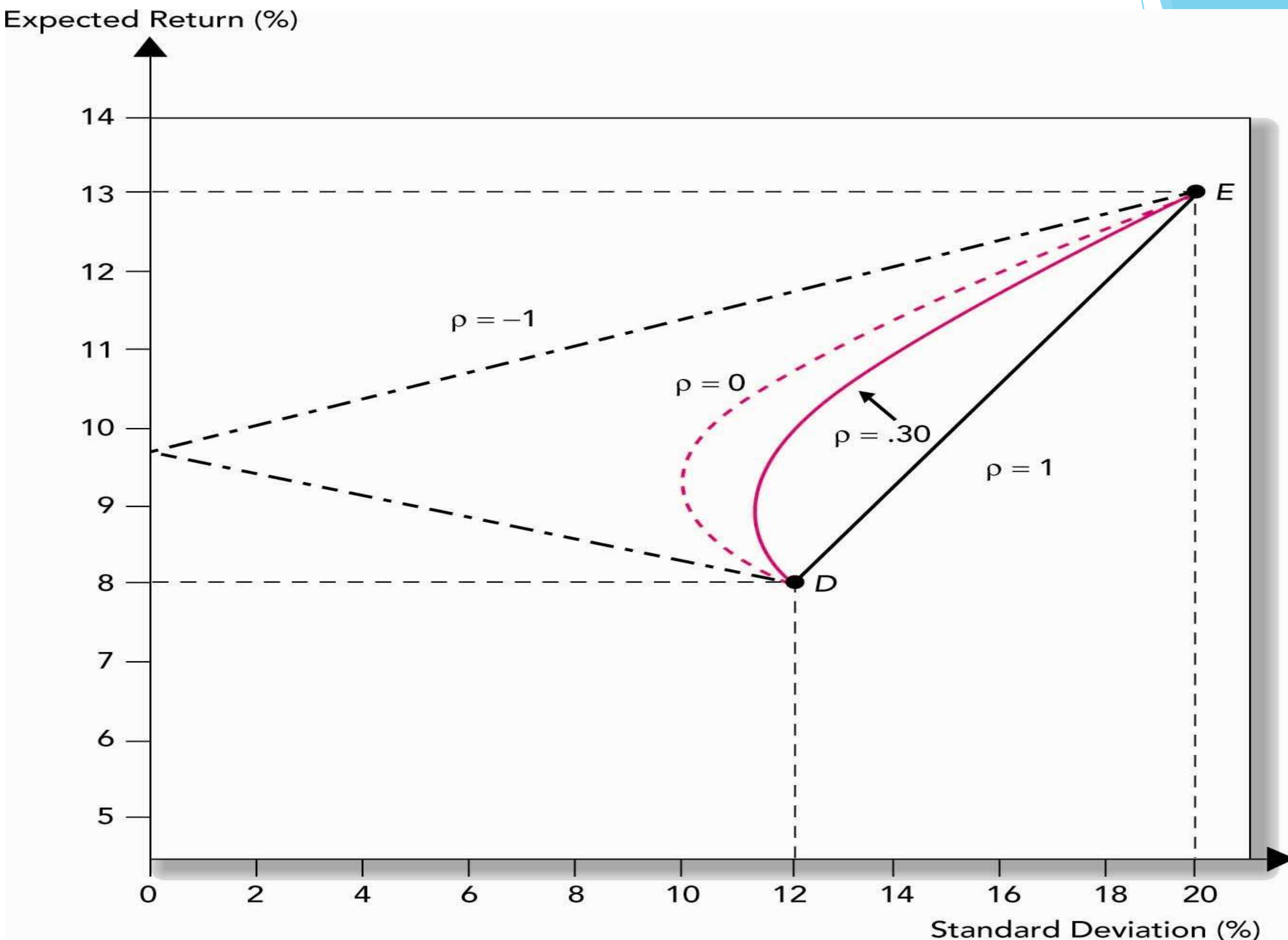
Correlation coefficient

- Correlation coefficient measures the mutual dependence of two random returns.
- Correlation coefficient ranges from +1 (perfectly positively correlated) to -1 (perfectly negatively correlated).
- Cov(IBM,H1) = $p_1 * (E(r_{IBM}) - r_{IBM,\,1}) * (E(r_{H1}) - r_{H1,\,1}) + p_2 * (E(r_{IBM}) - r_{IBM,\,2}) * (E(r_{H1}) - r_{H1,\,2}) + \ldots + p_S * (E(r_{IBM}) - r_{IBM,\,S}) * (E(r_{H1}) - r_{H1,\,S})$.
- $\rho_{IBM,\,H1}$ = Cov(IBM,H1) / (Std(IBM) * Std(H1)).

2-asset diversification

		IBM	H1	H2
State	Prob.		r	
S1: cold	0.25	-0.1	0	0.1
S2: normal	0.5	0.1	0.05	0.05
S3: hot	0.25	0.3	0.1	0
E(r)		0.1	0.05	0.05
Var(r)		0.02	0.0013	0.0013
Std(r)		0.1414	0.0354	0.0354
Cov with IBM			0.005	-0.005
ρ with IBM			1	-1

2-asset diversification



So, these are what we have so far:

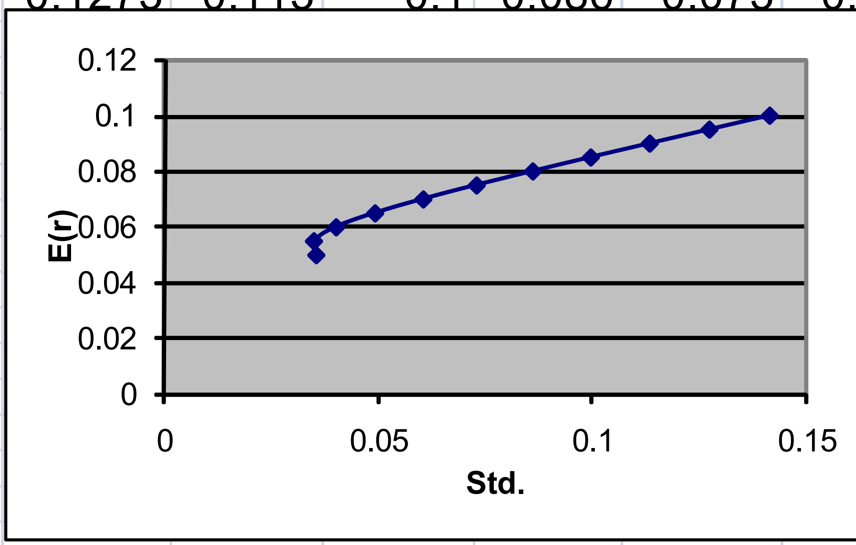
- All portfolios (with nonnegative weights) made from 2 assets lie on or to the left of the line connecting the 2 assets.
- The collection of the resulting portfolios is called the feasible set.
- Convexity (to the left): given any 2 points in the feasible set, the straight line connecting them does not cross the left boundary of the feasible set.

2-asset formulas

- It also turns out that there are nice formulas for calculating the expected return and standard deviation of a 2-asset portfolio. Let the portfolio weight of asset 1 be *w*. The portfolio weight of asset 2 is thus $(1 - w)$.
- $E(r) = w * E(r_1) + (1 - w) * E(r_2)$.
- $Std(r) = (w^2 * Var(r_1) + 2 * w * (1 - w) * cov(1,2) + (1 - w)^2 * Var(r_2))^{1/2}$.

Now, let us work on ρ = 0, i.e., Cov=0

	E(r)	Var(r)	Cov ()						
IBM	0.1	0.02							
H3	0.05	0.0013	0						
IBM	1	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2
H3	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
E(r)	0.1	0.095	0.09	0.085	0.08	0.075	0.07	0.07	0.06
Std(r)	0.141	0.1273	0.113	0.1	0.086	0.073	0.06	0.05	0.04
	0.1						0.07	0.065	0.06

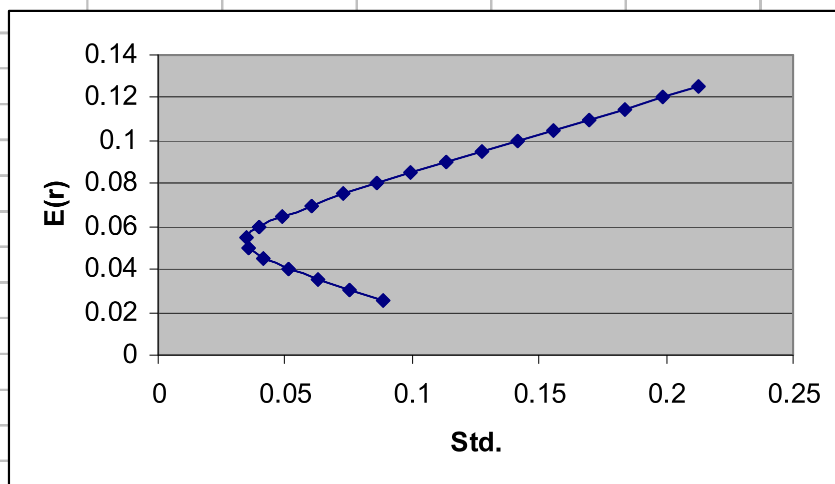


What if one can short sell?

- The previous calculations do not use negative weights; that is, short sales are not considered.
- This is usually the case in real-life institutional investing.
- Many institutions are forbidden by laws from short selling; many others self-impose this constraint.
- When short sales are allowed, the opportunity set expands. That is, more mean-variance combinations can be achieved.

Short sales allowed, $\rho = 0$

	E(r)	Var(r)	Cov ()						
IBM	0.1	0.02							
H3	0.05	0.001	0						
IBM	1.5	1.4	1.3	1.2	1.1	1	0.9	0.8	0.7
H3	-0.5	-0.4	-0.3	-0.2	-0.1	0	0.1	0.2	0.3
E(r)	0.125	0.12	0.115	0.11	0.105	0.1	0.095	0.09	0.085
Std(r)	0.213	0.198	0.184	0.17	0.156	0.141	0.127	0.113	0.1
	0.125	0.12	0.115	0.11	0.105	0.1	0.095	0.09	0.085



The general properties of combining 2 assets

- In real life, the correlation coefficient between 2 assets has an intermediate value; you do not see +1 and -1, particularly -1.
- When the correlation coefficient has an intermediate value, we can use the 2 assets to form an infinite combination of portfolios.
- This combination looks like the curve just shown.
- This curve passes through the 2 assets.
- This curve has a bullet shape and has a left boundary point, i.e., convexity.

N>2 assets, I

- When we have a large number of assets, what kind of feasible set can we expect?
- It turns out that we still have convexity: given any 2 assets (portfolios) in the feasible set, the straight line connecting them does not cross the left boundary of the feasible set.
- When all combinations of any 2 assets (portfolios) have this property, the left boundary of the feasible set also has a bullet shape.

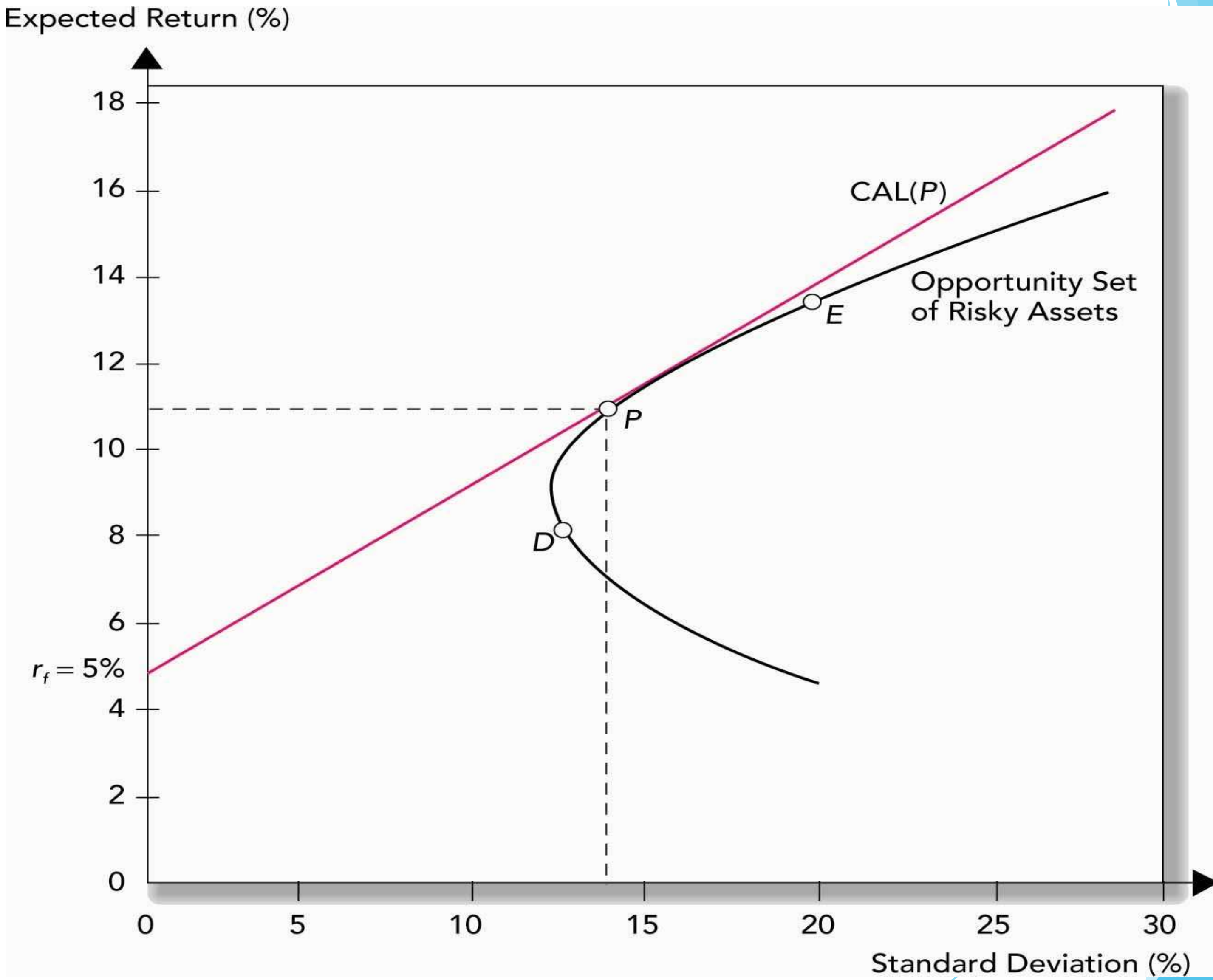
N>2 assets, II

- The left boundary of a feasible set is called the minimum-variance frontier (MVF) because for any value of expected return, the feasible point with the smallest variance (std.) is the corresponding left boundary point.
- The point on the minimum-variance frontier that has the minimum variance is called the minimum-variance point (MVP). This point defines the upper and the lower portion of the minimum-variance set.
- The upper portion of the minimum-variance set is called the efficient frontier (EF).

N>2 assets, III

- Only the upper part of the mean-variance frontier, i.e., the efficient frontier, will be of interest to investors who are:
- (1) risk averse: holding other factors constant, the lower the variance (std), the better, and
- (2) non-satiation: holding other factors constant, the higher the expected return, the better.

N-asset diversification



Selecting an optimal portfolio from N>2 assets

- An **optimizer** (using quadratic programming) is used to identifying the set of permissible optimal portfolios.
- Given the efficient frontier (EF), selecting an optimal portfolio for an investor who are allowed to invest in a combination of N risky assets is rather straightforward.
- One way is to ask the investor about the comfortable level of standard deviation (risk tolerance), say 20%. Then, corresponding to that level of std., we find the optimal portfolio on the EF, say the portfolio E shown in the previous figure.

What if one can invest in the risk-free asset?

- So far, our discussions on N assets have focused only on risky assets.
- If we add the risk-free asset to N risky assets, we can enhance the efficient frontier (EF) to the red line shown in the previous figure, i.e., the straight line that passes through the risk-free asset and the tangent point of the efficient frontier (EF).

Enhanced efficient frontier

- With the risk-free asset, the red straight line will be of interest to investors who are (1) risk averse, and (2) non-satiation.
- Why the red line pass through the tangent point? The reason is that this line has the highest slope; that is, given one unit of std. (variance), the associated expected return is the highest; consistent with the preferences in (1) and (2).
- Why the red line is a straight line? This is because the risk-free asset, by definition, has zero variance (std.) and zero covariance with any risky asset.

Portfolio theory in real life, I

- Portfolio theory is probably the mostly used modern financial theory in practice.
- The foundation of asset allocation in real life is built on portfolio theory.
- Asset allocation is the portfolio optimization done at the asset class level. An asset class is a group of similar assets.
- Virtually every fund sponsor in U.S. has an asset allocation plan and revises its (strategic) asset allocation annually.

Portfolio theory in real life, II

- Most fund sponsors do **not** short sell.
- They often use a quadratic program to generate the efficient frontier (or enhanced efficient frontier) and then choose an optimal portfolio on the efficient frontier (or enhanced efficient frontier).
- See my hand-out for quadratic programming.
- Many commercial computer packages, e.g., Matlab, have a built-in function for quadratic programming.
- This calculation requires at least two sets of inputs (estimates): expected returns and covariance matrix of asset classes.
- The outputs from the optimization include portfolio weights for asset classes. Asset allocation is based on these optimized portfolio weights.

A typical asset allocation: sign-constrained minimum-variance frontier

- Most fund sponsors require the asset-class weights be non-negative and sum to 1.
- Corner portfolios arise because these non-negative constraints on asset-class weights.
- Corner portfolios are special portfolios to the portfolio optimization problem in which the quantity of one of the asset class weights in the maximized function is zero.

Corner portfolio theorem

- (Two) adjacent corner portfolios define a segment of the minimum-variance frontier within which (1) portfolios hold identical assets, and (2) the rate of change of asset weights in moving from one portfolio to another is constant.

- Implication: all minimum-variance portfolios within the inside segment can be created by a linear combination of the two outside adjacent corner portfolios.

Thank You