PHYS 3014: Statistical Mechanics

Lecture Notes
Part 17

Bose Gas : Equation of State

Programme: B. Sc. Physics
Semester: VI

Dr. Ajai Kumar Gupta
Professor
Department of Physics
Mahatma Gandhi Central University
Motihari-845401, Bihar
E-mail: akgupta@mgcub.ac.in

Equation of State

for Bose gas

$$\frac{P}{KT} = -\frac{2\pi}{h^3}(2m)^{3/2}\int_0^\infty \epsilon^{1/2}\ln(1-ze^{-\beta\epsilon})\,d\epsilon - \frac{1}{V}\ln(1-z) \quad \text{---} (1)$$

and.

$$\frac{N}{V} = \frac{2\pi}{h^3}(2m)^{3/2}\int_0^\infty \frac{\epsilon^{1/2}\,d\epsilon}{z^{-1}e^{\beta\epsilon}-1} + \frac{1}{V}\frac{z}{1-z} \quad \text{---} (2)$$

If $z<1$, in eqn (1), $\ln(1-z)$ is finite and $\frac{1}{V}\ln(1-z)$ becomes zero when $V\to\infty$. When $z=1$, then $\frac{z}{1-z}\simeq N$ or, $1-z\simeq\frac{1}{1+N}$. Therefore $-\frac{1}{V}\ln(1-z)\simeq\frac{1}{V}\ln\left(\frac{1}{1-z}\right)$ $=\frac{1}{V}\ln(1+N)$. vanishes for $N\to\infty$ and $V\to\infty$. So for all values of z, $-\frac{1}{V}\ln(1-z)$ is negligible.

So equ^n ① becomes

$$\frac{P}{KT} = -\frac{2\pi}{h^3}(2\pi m K)^{3/2}\int_0^\infty x^{1/2}\ln(1-z e^{-x})\,dx \quad \because \beta\epsilon = x$$

Using Bose - Einstein function

$$\frac{P}{KT} = +\frac{2\pi}{h^3}(2\pi m K)^{3/2}\cdot\frac{2}{3}\int_0^\infty x^{3/2}\frac{dx}{z^{-1}e^{x}-1}$$

or,

$$\frac{P}{KT} = \frac{(2\pi m KT)^{3/2}}{h^3}\cdot g_{5/2}(z) \quad - - - - ③$$

$$= \frac{1}{\lambda^3}g_{5/2}(z)$$

and

$$\frac{N-N_0}{V} = \frac{1}{\lambda^3}g_{3/2}(z) \quad - - - ④$$

By eliminating z from equ^n ③ and ④ we obtain the equation of state of the system.

Below the critical temperature $T < T_c$, $z = 1$

and pressure becomes

$$P = \frac{KT}{\lambda^3} \zeta\left(\frac{5}{2}\right) \quad \text{---} \quad (5) \qquad P \propto T^{5/2}$$

It is independent of volume and particle number and only depends on the temperature. It is also because particles in the ground state do not contribute to the pressure. If we add another particle to the system at $T < T_c$, particle will go to state $\epsilon = 0$ and do not contribute to pressure.

We can define a critical density at given temperature above which Bose Condensation occurs is

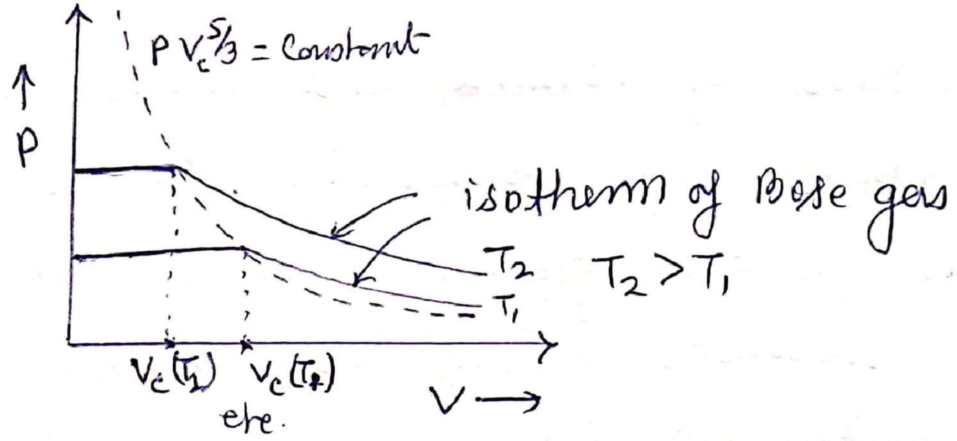
$$\left(\frac{N}{V}\right)_c = \frac{\zeta\left(\frac{3}{2}\right)}{\lambda^3}$$

and critical volume $V_c = \dfrac{N\lambda^3}{\zeta(3/2)}$ --- (6)

$$V_c \propto \frac{1}{T^{3/2}}$$

From eqn (5) and (6) eliminating temperature T, we get

$$P V_c^{5/3} = \text{Constant} \quad --- (7)$$



At $T = T_c$,

$$P(T_c) = \frac{\zeta(\frac{5}{2})}{\zeta(\frac{3}{2})} \cdot \left(\frac{N}{V} K T_c\right) \simeq 0.5134 \left(\frac{N}{V} \cdot K \cdot T_c\right) \quad -- (8)$$

pressure exerted by particles of an ideal Bose gas at the transition temperature T_c is about one half of that exerted by particles of a Boltzmann gas.

For $T > T_c$, pressure is given by

$$P = \frac{N}{V} KT \frac{g_{5/2}(z)}{g_{3/2}(z)} \quad \text{----} \ (9)$$

and $z(T)$ can be determined by equation

$$g_{3/2}(z) = \frac{N}{V} \cdot \frac{h^3}{(2\pi mKT)^{3/2}} = \frac{N}{V} \lambda^3 \quad \text{--} \ (10)$$

In condensed state $T < T_c$, pressure is independent of volume, it means that isotherms of Bose gas are horizontal lines in the PV diagram for volumes $V < V_c$.

Bose condensation is a phase transition in a system. One phase is represented by the particles in the excited states and other phase is formed by particles in the state $\epsilon = 0$.

For $T < T_c$

$$\frac{P(T)}{P(T_c)} = \frac{KT}{\lambda^3} \cdot \zeta\left(\frac{5}{2}\right) \cdot \frac{\zeta\left(\frac{3}{2}\right)}{\zeta(5/2)} \cdot \frac{1}{\left(\frac{N}{V} K T_c\right)}$$

$$= T^{5/2} \cdot \frac{(2\pi m k)^{3/2}}{h^3} \cdot \zeta\left(\frac{3}{2}\right) \cdot \frac{h^3}{(2\pi m k)^{3/2} \cdot T_c^{3/2} \cdot T_c \cdot \zeta(3/2)}$$

$$\because \quad T_c = \frac{h^2}{2mk\pi}\left[\frac{N}{V\zeta(3/2)}\right]^{2/3}$$

or $$\boxed{\frac{P(T)}{P(T_c)} = \left(\frac{T}{T_c}\right)^{5/2}} \quad \text{—— (11)}$$

At classical limit $z \ll 1$, equ^n (9) Converts to

$\therefore \quad P = \frac{N}{V} KT \quad$ ideal gas law

Internal Energy of system

$$E = -\frac{\partial}{\partial \beta} \ln Z = KT^2 \left\{\frac{\partial}{\partial T}\left(\frac{PV}{KT}\right)\right\}_{z,V}$$

$$= KT^2 \left[\frac{\partial}{\partial T}\left(V \cdot \frac{1}{\lambda^3} g_{5/2}(z)\right)\right]_{z,V} \quad \text{from equ}^n \text{ (3)}$$

$$E = KT^2 V g_{5/2}(z) \frac{\partial}{\partial T}\left(\frac{1}{\lambda^3}\right)$$

$$= \frac{3}{2} KT \frac{V}{\lambda^3} g_{5/2}(z)$$

$$\therefore \quad \frac{PV}{KT} = \frac{2}{3}\frac{E}{KT}$$

$$\text{or,} \quad \boxed{P = \frac{2}{3}\frac{E}{V}} \quad \text{--- (12)}$$

Entropy of Bose gas

$$E = TS - PV + \mu N$$

For $T < T_c$, $\mu = 0$

$$S = \frac{E + PV}{T} = \frac{5}{2}\left(\frac{PV}{T}\right) \qquad \because E = \frac{3}{2} PV$$

$$\therefore \quad \frac{S}{NK} = \frac{5}{2} \cdot \frac{V}{N} \frac{1}{\lambda^3} \zeta\left(\frac{5}{2}\right)$$

using eqn (5)

$$= \frac{5}{2} \cdot \frac{\zeta(\frac{5}{2})}{\zeta(\frac{3}{2})} \left(\frac{T}{T_c}\right)^{3/2} = \frac{5}{2} \times \frac{1.341}{2.612} \left(\frac{T}{T_c}\right)^{3/2} = \frac{5}{2} \times .5134 \left(\frac{T}{T_c}\right)^{3/2}$$

Again $S = \frac{5}{2} K V \frac{1}{\lambda^3} \zeta\left(\frac{5}{2}\right)$

we know that $N_{ex} = \frac{V}{\lambda^3} \cdot \zeta\left(\frac{3}{2}\right)$

$$\therefore \quad S = \frac{5}{2} K N_{ex} \cdot \frac{\zeta\left(\frac{5}{2}\right)}{\zeta\left(\frac{3}{2}\right)}$$

Therefore in the condensed state N_0 particles do not contribute towards entropy of the system whereas N_{ex} particles · the normal part contribute

$\frac{5}{2} K \times \frac{1.341}{2.612}$ per particle.

References:

- Statistical Mechanics by R. K. Pathria
- Statistical Mechanics by B. K. Agarwal and M. Eisner
- An Introductory Course of Statistical Mechanics by P. B. Pal
- Elementary Statistical Physics by C. Kittel
- Fundamentals of Statistical and Thermal Physics by F. Reif
- Statistical and Thermal Physics by R. S. Gambhir and S. Lokanathan

Thank You

For any questions/doubts/suggestions and submission of assignments

Write at E-mail: akgupta@mgcub.ac.in