Lecture-IX Programme: M. Sc. Physics

Retarded Potentials and Fields

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Contents

- In electrodynamics, the ***retarded potentials*** are the electromagnetic potentials for the electromagnetic field produced by time-varying electric current or charge distributions in the past.
- The fields propagate with the speed of light c, so the delay of the fields linking reason and effect at earlier and later times is an significant feature: the signal takes definite time to propagate from a point in the charge or current distribution (the point of cause) to a different point in space (where the effect is detected).

Expressions for Retarded Potentials

❖ As we know that from previous study the Maxwell's equations in terms of scalar and vector potentials those are-

(i) $\Box^2 V = -\frac{1}{\epsilon_0}\rho,$ ------------- [1]

(ii) $\Box^2 \mathbf{A} = -\mu_0 \mathbf{J}.$ ------------- [2]

where $\nabla^2 - \mu_0\epsilon_0 \frac{\partial^2}{\partial t^2} \equiv \Box^2,$ is called the D' Alembertian operator

------------- [3]

❖ In the static case, equations 1 and 2 convert into (four copies of)

Poisson's equation-

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho$$ ------------- [4]

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$$ ------------- [5]

with the well known solutions

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}}\,d\tau'$$ ------------- [6]

and

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}}\,d\tau'$$ ------------- [7]

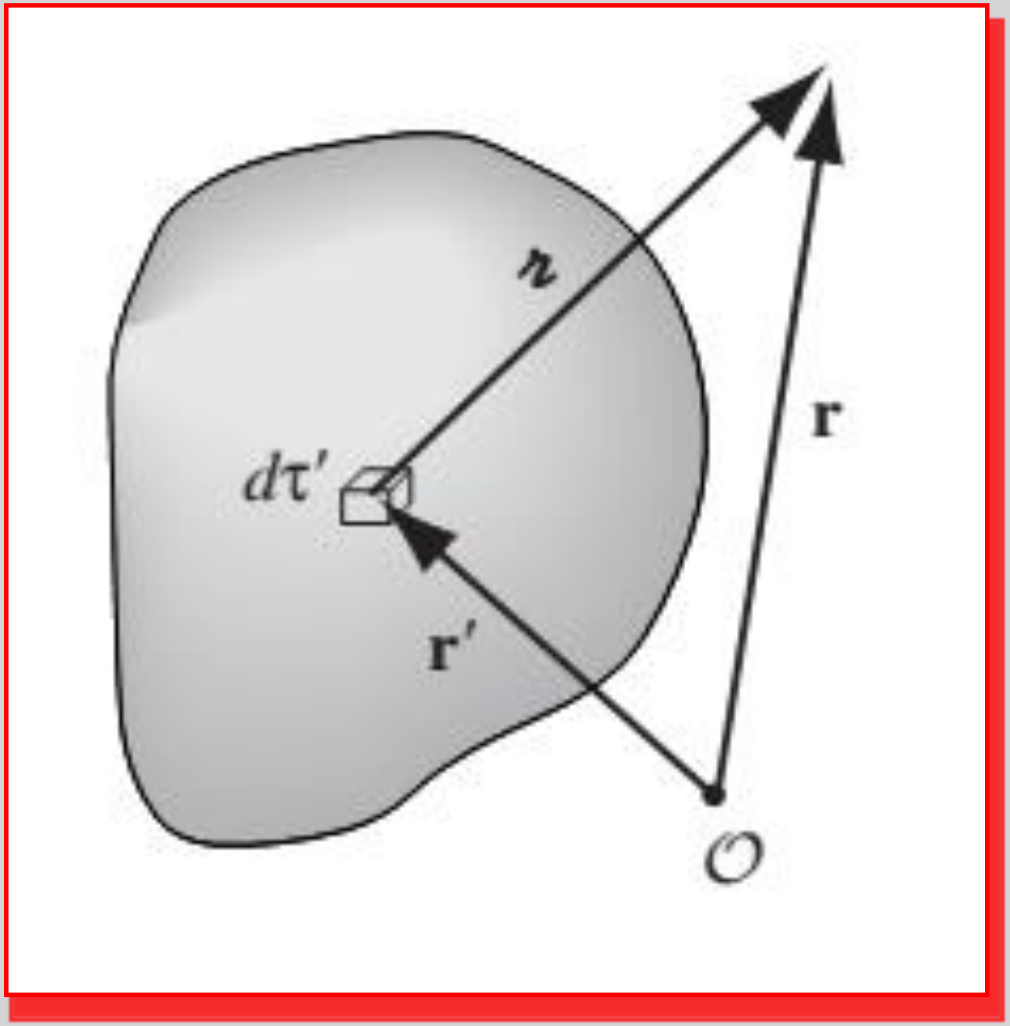


Figure 1: Continuous distribution of charge in an arbitrary volume [REF-1]**

where $\mathscr{r}$, as constantly, is the distance from the source point r' to the field point r (Fig. 1).

- **Since, electromagnetic "information" moves with speed of light. In the *non-static* situation, thus, it is not the status of the source right now that subjects, but somewhat its situation at some earlier time t_r (is known as the *retarded time*) when the "message" left. As this message must move a distance $\mathscr{r}$, the delay is therefore $\mathscr{r}/c$:**

$$t_r \equiv t - \frac{\mathscr{r}}{c}. \qquad \text{-------------} \quad [8]$$

❖ Since the integrals are calculated at the retarded time. Therefore they are known as retarded potentials.

❖ Note that the retarded potentials can reduce into suitably to in the static case defined by the equations (6) and (7), for which ρ and J are should be independent of time.

❖The normal simplification of equations (6) and (7) for retarded case (dynamic sources), is thus-

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t_r)}{\mathscr{r}} d\tau' \qquad \text{-------------- [9]}$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t_r)}{\mathscr{r}} d\tau' \qquad \text{-------------- [10]}$$

where $\rho(r', t_r)$ is the charge density that prevail at point r' at the retarded time t_r.

Jefimenko's Equations

- Given that retarded potentials from previous study-

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t_r)}{\mathscr{r}} d\tau',$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t_r)}{\mathscr{r}} d\tau',$$

- To find the expressions for electric and magnetic fields, we will use the following identities:

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}.$$

-------------- [11]

❖ Although the aspects are not completely insignificant because, as we talked about before, the integrands depend on r both explicitly, through $\mathscr{r} = |r - r'|$ in the denominator, and completely, through the retarded time $t_r = t - \mathscr{r}/c$ in the argument of the numerator.

❖The key point to see is that the integrand in equations 9 and 10 depend on r in two positions: clearly, in the denominator ($\mathscr{r} = |r - r'|$), and implicitly, through $t_r = t - \mathscr{r}/c$, in the numerator. Hence-

$$\nabla V = \frac{1}{4\pi\epsilon_0}\int\left[(\nabla\rho)\frac{1}{\mathscr{r}} + \rho\nabla\left(\frac{1}{\mathscr{r}}\right)\right]d\tau', \qquad \text{-------------- [12]}$$

$$\nabla \rho = \dot{\rho}\, \nabla t_r = -\frac{1}{c} \dot{\rho}\, \nabla \mathscr{r}$$ -------------- [13]

(The dot represents differentiation with respect to time). Now $\nabla \mathscr{r} = \hat{\mathscr{r}}$

$\nabla(1/\mathscr{r}) = -\hat{\mathscr{r}}/\mathscr{r}^2$

so
$$\nabla V = \frac{1}{4\pi\epsilon_0} \int \left[-\frac{\dot{\rho}}{c} \frac{\hat{\mathscr{r}}}{\mathscr{r}} - \rho \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right] d\tau'.$$ -------------- [14]

Now the time derivative of vector potential A can be defined:

$$\frac{\partial \mathbf{A}}{\partial t} = \frac{\mu_0}{4\pi} \int \frac{\dot{\mathbf{J}}}{\mathscr{r}}\, d\tau'.$$ -------------- [15]

Substituting these in equation (11) (and using $c^2 = 1/\mu_0\varepsilon_0$) and simplify and we obtain:

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \left[\frac{\rho(\mathbf{r}', t_r)}{\mathscr{r}^2} \hat{\mathscr{r}} + \frac{\dot{\rho}(\mathbf{r}', t_r)}{c\mathscr{r}} \hat{\mathscr{r}} - \frac{\dot{\mathbf{J}}(\mathbf{r}', t_r)}{c^2\mathscr{r}} \right] d\tau'. \qquad \text{---------- [16]}$$

❖ The equation (16) is representing the ***time-dependent generalization of Coulomb's law***, it can be turn into the static case (where the second and third terms leave and the first term loses its dependence on t_r).

❖ Expression for B, the curl of A contains two terms:

$$\nabla \times \mathbf{A} = \frac{\mu_0}{4\pi} \int \left[\frac{1}{\mathscr{r}} (\nabla \times \mathbf{J}) - \mathbf{J} \times \nabla \left(\frac{1}{\mathscr{r}} \right) \right] d\tau'. \qquad \text{-------------- [17]}$$

Now

$$(\nabla \times \mathbf{J})_x = \frac{\partial J_z}{\partial y} - \frac{\partial J_y}{\partial z}, \qquad \text{------------- [18]}$$

and

$$\frac{\partial J_z}{\partial y} = \dot{J}_z \frac{\partial t_r}{\partial y} = -\frac{1}{c}\dot{J}_z \frac{\partial \mathscr{r}}{\partial y}, \qquad \text{------------- [19]}$$

therefore

$$(\nabla \times \mathbf{J})_x = -\frac{1}{c}\left(\dot{J}_z \frac{\partial \mathscr{r}}{\partial y} - \dot{J}_y \frac{\partial \mathscr{r}}{\partial z}\right) = \frac{1}{c}\left[\dot{\mathbf{J}} \times (\nabla \mathscr{r})\right]_x .$$

But $\nabla \mathscr{r} = \hat{\mathscr{r}}$, so

$$\nabla \times \mathbf{J} = \frac{1}{c}\dot{\mathbf{J}} \times \hat{\mathscr{r}}. \qquad \text{------------- [20]}$$

As we know that $\nabla(1/\mathscr{r}) = -\hat{\mathscr{r}}/\mathscr{r}^2$, and therefore

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \left[\frac{\mathbf{J}(\mathbf{r}', t_r)}{\mathscr{r}^2} + \frac{\dot{\mathbf{J}}(\mathbf{r}', t_r)}{c\mathscr{r}} \right] \times \hat{\mathscr{r}} \, d\tau'. \quad \text{-------------- [21]}$$

❖ This is the ***time-dependent generalization of the Biot-Savart law***, to which it reduces in the static case.

Equations 16 and 21 are the fundamental solutions to Maxwell's equations in time dependent situation.

❖ **For some basis, they do not look to have been published until rather recently- the earliest clear statement given by Oleg Jefimenko, in 1966.**

❖ **In practice *Jefimenko's* equations are of limited usefulness, as it is naturally easier to determine the retarded potentials and differentiate them, rather than going directly to the fields. But, they offer a fulfilling sense of conclusion to the theory.**

❖They also assist to simplify an study we finished in the earlier segment: To obtain to the ***retarded potentials***, all you do is change *t* by t_r in the electrostatic and magneto-static relations, however in the case of the *fields* not only is time replaced by retarded time, but entirely novel terms (linking derivatives of ρ and J) appear. And they offer amazingly great support for the quasi-static approximation.

Numerical problems

1. In a region where $\mu_r = \varepsilon_r = 1$ and $\sigma = 0$, the retarded potentials are given by $V = x(z - ct)$ volt and $A = x(z/c - z)\, a_z$ Wb/m, $c = 1/(\mu_0\varepsilon_0)^{1/2}$. (a) Show that $\nabla . A = \mu_0 \varepsilon\, dV/dt$. (b) Find B, H, E, and D (c) Show that these results satisfy Maxwell's equations if J and ρ_v are zero.

2. An infinite straight wire carries the current $I(t) = \begin{cases} 0, & \text{for } t \leq 0, \\ I_0, & \text{for } t > 0. \end{cases}$

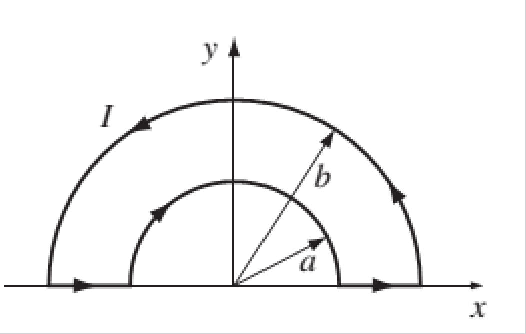
   That is, a constant current I_0 is turned on abruptly at $t = 0$. Find the resulting electric and magnetic fields.

3. Confirm that the retarded potentials satisfy the Lorenz gauge condition.

4. (a) Suppose the wire in see problem 2 carries a linearly increasing current $I(t) = kt$, for $t > 0$. Find the electric and magnetic fields generated (b) Do the same for the case of a sudden burst of current: $I(t) = q_0\delta(t)$.

5. A piece of wire bent into a loop, as shown in, carries a current that increases linearly with time: $I(t) = kt \quad (-\infty < t < \infty).$

Calculate the retarded vector potential A at the center. Find the electric field at the center. Why does this (neutral) wire produce an *electric* field? (Why can't you determine the *magnetic* field from this expression for A?)



References:

1. ***Introduction to Electrodynamics, David J. Griffiths***

2. ***Elements of Electromagnetics, 2nd edition by M N O Sadiku.***

3. ***Engineering Electromagnetics by W H Hayt and J A Buck.***

4. ***Elements of Electromagnetic Theory & Electrodynamics, Satya Prakash***

- **For any query/ problem contact me on whatsapp group or mail on me**

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- **Next *** we will discuss Retarded Potentials: Liénard - Wiechert Potentials and numerical problems of this lecture and.**

Thank you