PHYS 3014: Statistical Mechanics

Lecture Notes
Part 20

Fermi Gas : Completely & Strongly degenerate Cases

Programme: B. Sc. Physics
Semester: VI

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Completely Degenerate Fermi Gas

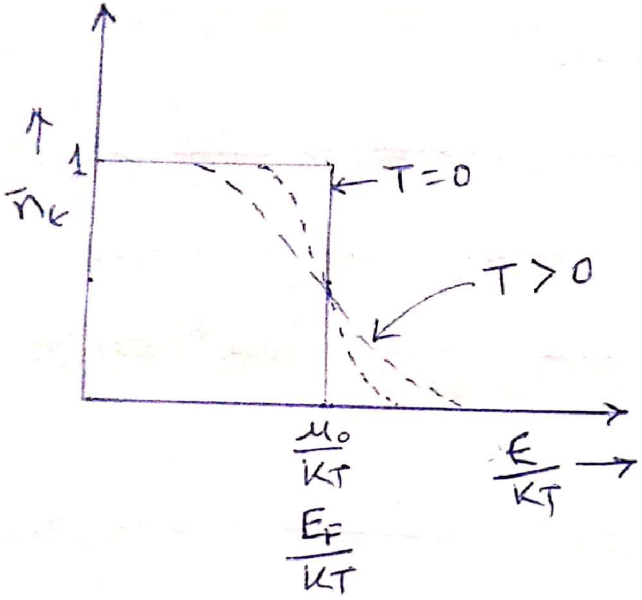
At sufficiently low temperatures $\frac{N\lambda^3}{Vg} >> 1$ and gas becomes degenerate. when $\frac{N\lambda^3}{Vg} \to \infty$ which corresponds to $T = 0$, the gas is called as completely degenerate.

In the limit $T \to 0$ or $\frac{N\lambda^3}{Vg} \to \infty$, the mean occupation number of the single particle state ϵ is

$$\bar{n}_\epsilon = \frac{1}{e^{\frac{\epsilon - \mu}{KT}} + 1} = 1 \quad \text{for } \epsilon < \mu_0$$
$$= 0 \quad \text{for } \epsilon > \mu_0$$

where μ_0 is the chemical potential at 0 K.

$\bar{n}_\epsilon$ is step function.



At $T=0$, all single particle states upto $\epsilon = \mu_0$ are completely filled with one particle per state according to Pauli's exclusion principle. The single particle states with $\epsilon > 0$ are empty. This limiting value of the chemical potential μ_0 is called as Fermi energy of the system and is denoted by E_F. The corresponding single particle momentum is called as Fermi momentum P_F.

$$E_F(0) = \mu_0 = \frac{P_F^2}{2m}$$

$E_F(0)$ represents the energy of the highest occupied level at 0 K. As temperature increases above 0 K, the distribution near $E_F(0)$ rounds off. We can also define a temperature called as Fermi temperature T_F by

$$T_F = \frac{E_F(0)}{K}$$

In terms of Fermi temperature T_F, for $T << T_F$ or $KT << E_F(0)$, the gas is called degenerate and for $T >> T_F$, gas is called non-degenerate and we get classical result.

When T increases from $T = 0$ K, particles are excited from single particle states with energy near Fermi energy $\epsilon < E_F$ to single particle states with energy $\epsilon > E_F$. The thermal excitation of the particles occurs only in a narrow energy range around E_F and has the width of the order of KT. The fraction of the particles which are thermally excited is of the order of $\frac{KT}{E_F(0)}$. The major part of system remains unaffected by the rise in temperature.

At 0 K, the no. of particles in the system

$$N = \int_0^{E_F(0)} g(\epsilon)\, d\epsilon \quad , \quad g(\epsilon) = g \frac{2\pi V}{h^3} (2m)^{3/2} \epsilon^{1/2}$$

is the no. of states per unit energy range of the system.

$$\therefore \quad N = \frac{4\pi g V}{3h^3} E_{F(0)}^{3/2} (2m)^{3/2} \quad \text{---} \; (1)$$

$$\therefore \quad E_F(0) = \left(\frac{3N}{4\pi g V}\right)^{2/3} \left(\frac{h^2}{2m}\right) \quad \text{--- (2)}$$

$$P_F(0) = \left(\frac{3N}{4\pi g V}\right)^{1/3} h \quad \text{--- (3)}$$

$$T_F = \frac{E_F(0)}{k} \quad \text{--- (4)}$$

The expectation value of energy or ground state energy or zero point energy of the system is given by

$$E_0 = \frac{2\pi g V}{h^3} (2m)^{3/2} \int_0^{E_F(0)} \epsilon \cdot \epsilon^{1/2} \, d\epsilon$$

$$= \frac{2}{5} \cdot \frac{2\pi g V}{h^3} (2m)^{3/2} \cdot \left[E_F(0)\right]^{5/2} \quad \text{--- (5)}$$

$$\therefore \quad \frac{E_0}{N} = \text{average energy per particle at } 0K$$

$$= \frac{3}{5} E_F(0) \quad \text{--- (6)}$$

three by five times the Fermi energy at 0 K.

The ground state pressure of the system

$$P_0 = \frac{2}{3}\frac{E_0}{V}$$

$$= \frac{2}{5}\frac{N}{V}E_F(0) = \frac{2}{5}\frac{N}{V}\cdot\left(\frac{3N}{4g\pi V}\right)^{2/3}\frac{h^2}{2m} \quad \text{--- (7)}$$

$$P_0 \propto \left(\frac{N}{V}\right)^{5} \quad \text{--- (8)}$$

Strongly degenerate Fermi gas $T \ll T_F$ but $T \neq 0$.

At finite low temperatures, the value of fugacity z is finite but large in comparison to unity i.e $z > 1$.

The total number of particles and total energy of system is

$$N = \frac{Vg}{\lambda^3} f_{3/2}(z) \quad \text{--- (9)}$$

and
$$U = \frac{3}{2}KT\frac{gV}{\lambda^3} f_{5/2}(z) = \frac{3}{2}NKT\frac{f_{5/2}(z)}{f_{3/2}(z)} \quad \text{--- (10)}$$

for $z > 1$ (positive μ)

to the first approximation

$$f_{5/2}(z) = \frac{8}{15\pi^{1/2}} (\ln z)^{5/2} \left[1 + \frac{5\pi^2}{8} (\ln z)^{-2} + \cdots \right] \quad \text{--- (11)}$$

$$f_{3/2}(z) = \frac{4}{3\pi^{1/2}} (\ln z)^{3/2} \left\{ 1 + \frac{\pi^2}{8} (\ln z)^{-2} + \cdots \right] \quad \text{--- (12)}$$

and

$$f_{1/2}(z) = \frac{2}{\pi^{1/2}} (\ln z)^{1/2} \left[1 - \frac{\pi^2}{24} (\ln z)^{-2} + \cdots \right] \quad \text{--- (13)}$$

$$\therefore \quad \frac{N}{V} = \frac{4\pi g}{3} \left(\frac{2m}{h^2} \right)^{3/2} (KT \ln(z))^{3/2} \left[1 + \frac{\pi^2}{8} (\ln z)^{-2} + \cdots \right] \quad \text{--- (14)}$$

In the zeroth approximation

$$KT \ln(z) = \left(\frac{3N}{4\pi g V} \right)^{2/3} \frac{h^2}{2m}$$

or, $$\mu = \left(\frac{3N}{4\pi g V} \right)^{2/3} \frac{h^2}{2m} \quad \text{--- (15)}$$

Equal to the value of $\mu_0 = E_F(0)$

In the next approximation

$$KT\ln(z) = \left(\frac{3N}{4\pi g V}\right)^{2/3}\frac{h^2}{2m}\cdot\left[1+\frac{\pi^2}{8}(\ln z)^{-2}\right]^{-2/3}$$

or, $$\mu \simeq E_F(0)\left[1-\frac{\pi^2}{12}\left(\frac{KT}{E_F(0)}\right)^2\right] \quad \because KT\ln z = E_F(0) \quad \text{---} (16)$$

and $$\frac{U}{N} = \frac{3}{2}KT\frac{f_{5/2}(z)}{f_{3/2}(z)} \simeq \frac{3}{2}KT\cdot\frac{2}{5}\ln(z)\cdot\frac{\left\{1+\frac{5\pi^2}{8}(\ln z)^{-2}+\cdots\right\}}{\left\{1+\frac{\pi^2}{8}(\ln z)^{-2}+\cdots\right\}}$$

$$\simeq \frac{3}{5}KT\ln(z)\cdot\left[1+\frac{\pi^2}{2}(\ln z)^{-2}+\cdots\right] \quad \because (1+x)^{-1}\simeq 1-x$$

$$\therefore \frac{U}{N} = \frac{3}{5}E_F(0)\left[1+\frac{5\pi^2}{12}\left(\frac{KT}{E_F(0)}\right)^2+\cdots\right] \quad \text{---} (17)$$

on substitution of value of $KT\ln(z)$ from eqn (16) and solving.

The pressure of the gas.

$$P = \frac{2}{3}\frac{U}{V} = \frac{2}{5}\frac{N}{V}E_F(0)\left[1+\frac{5\pi^2}{12}\left(\frac{KT}{E_F(0)}\right)^2+\cdots\right] \text{--} (18)$$

Internal energy of the system does not approach zero for low temperatures. It converges to a finite value.

The specific heat of the system

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V$$

$$= NK\left[\frac{\pi^2}{2}\left(\frac{KT}{E_F(0)}\right) - \frac{3\pi^4}{20}\left(\frac{KT}{E_F(0)}\right)^3 - \cdots\right]$$

or,

$$\frac{C_V}{NK} = \frac{\pi^2}{2}\left(\frac{T}{T_F}\right) - \frac{3\pi^4}{20}\left(\frac{T}{T_F}\right)^3 - \cdots \quad \text{--- (19)}$$

Thus for degenerate Fermi gas, specific heat varies as:

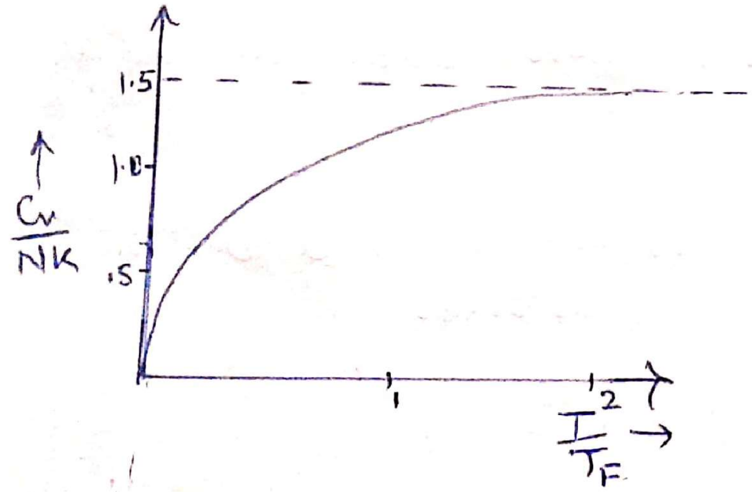
$$C_V = aT + bT^3 \quad \text{-- (20)}$$

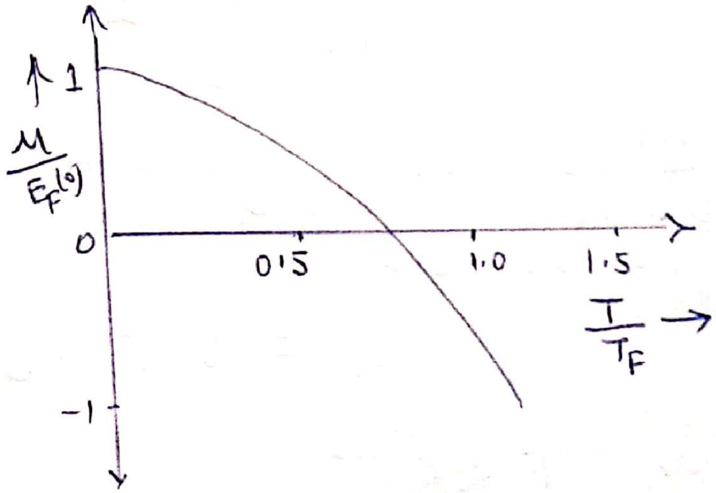
where a and b are constants.

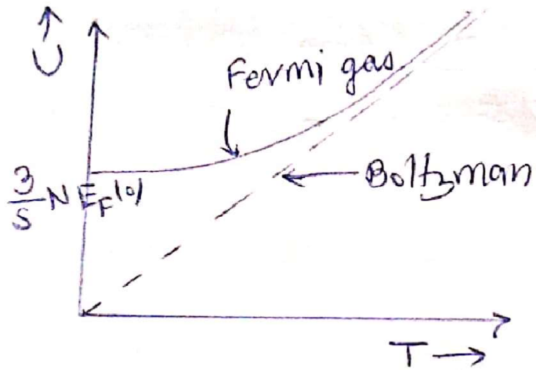
For $T << T_F$, $C_V = aT$ --- (21)

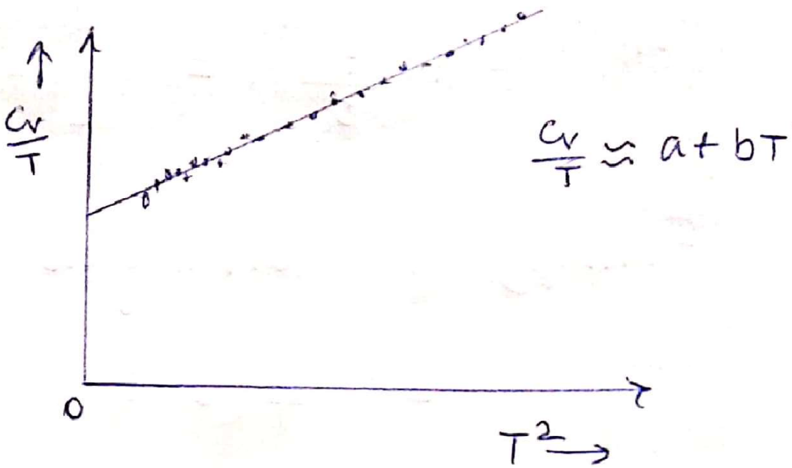
i.e $C_v \to 0$ as $T \to 0$. It is considerably smaller that the classical value $\frac{3}{2}NK$.

At low temperature the value of specific heat is very small. The contribution of electrons grows linearly with temperature. At the same time contribution of phonons are also there. (Debye Model $C_v \propto T^3$ at low temperature). But at very low temperature main contributions to the specific heat is from electrons.









In the plot of $\frac{C_V}{T}$ with T^2, after extrapolating the graph to $T=0$, the obtained intercept on y-axis will give the value of $\frac{\pi^2}{2} NK^2/E_F(0)$ which represents the electronic contribution to the specific heat whereas the slope of the straight line will give the contribution f [illegible] to specific heat of Fermi gas.

The Helmholtz free energy

$$F = \mu N - PV$$

or, $$F = \frac{3}{5} N E_F(0)\left[1 - \frac{5\pi^2}{12}\left(\frac{KT}{E_F(0)}\right)^2 + \cdots\right]$$

Entropy of Fermi gas

$$S = \frac{1}{T}(U-F) \quad , \quad S = \int \frac{C_V}{T}\, dT$$

$$= \frac{\pi^2}{2}\left(\frac{KT}{E_F(0)}\right) + \cdots$$

as $T \to 0$, $S \to 0$ (Third law), it means that the degenerate Fermi gas at $T = 0$ represents the state of system with the highest degree of order i.e perfect ordered system.

References:

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- Elementary Statistical Physics by C. Kittel
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Thank You

For any questions/doubts/suggestions and submission of assignments

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