Metrizability

There are many topological spaces which are not arises (induced) out of a metric. For example, a non-hausdorff space cannot be induced by a metric. Thus, it is interesting to know which spaces can be induced by a metric.

Let (X, d) be any metric space and $\mathcal{B}$ be the collection of all open balls in the metric space X. Let $T_d = \langle \mathcal{B} \rangle$, i.e., T_d is a topology on X generated by the base $\mathcal{B}$.

Defn: A top. space (X, T) is said to be metrizable if $\exists$ a metric 'd' on X s.t. $T = T_d$. In other words, we can say that a top. space X is metrizable if it can be induced by a metric on X.

Remark: Every metric space is a top. space but every top. space may not be metrizable.

We mention some results in the following:

1. Metrizability is heriditary, i.e., if X is a metrizable space and Y is a subspace of X, then Y is also metrizable.
2. If X is a discrete top. space, then it is metrizable.
3. If X is a finite set, then every metric on X induces th discrete top. on it.

Proof: The above results are easy to verify. Hence, I will leave it as an exercise. #

Remark: From the above result (3), it is clear that on a finite set the discrete topology is the only metrizable topology.

Lemma: Let (X, d) be a metric space and Y be a top. space. Suppose that $f : X \to Y$ is a homeomorphism (i.e, f is bijective and f & f^{-1} both are continuous). Then $d^* : Y \times Y \to \mathbb{R}$, defined by $d^*(y_1, y_2) = d(f^{-1}(y_1), f^{-1}(y_2))$, is a metric on Y and top. on Y is induced by the metric d^*.

Proof: (Outline only): With the help of metric property of d, it can be easily shown that d^* is a metric. Also, it can be observed that the above map f, when treated from $(X, d) \to (Y, d^*)$ will turns out to be an isometry (i.e., distance preserving map) and which implies that the top. on Y is induced by d^* #

Def^n: Let X be a T_1 space (i.e., finite sets are closed in X). Then X is said to be

(i) 'regular' if $\forall x \in X$ and $\forall$ closed set $F \subseteq X$ with $x \notin F$, $\exists$ disjoint open sets U & V in X s.t. $x \in U$ and $F \subseteq V$.

(ii) 'normal' if disjoint closed sets in X are separated by disjoint open sets of X. I.e., if A & B are closed in X with $A \cap B = \phi$, then $\exists$ disjoint open sets U & V of X such that $A \subseteq U$ and $B \subseteq V$.

Defn: A top. space X is said to be second countable if it has a countable base.

Theorem: If X is a metrizable space and $Y \cong X$, then Y is also metrizable.

Pf: Follows from the above Lemma. #

Theorem: (Urysohn Metrization Theorem) Every regular second countable space is metrizable.

Proof: (Outline only).

Using the assumptions that X is regular and second countable, it can be shown that X can be embedded in a metric space. Therefore, X is homeomorphic to a subspace of a metric space. Since a subspace of a metrizable space is metrizable and metrizability is preserved under a homeomorphism, it follows that X is metrizable. For detailed proof see [1].

Theorem (Bing Metrization Theorem): A top. space X is metrizable iff it is regular and has a basis that is countably locally discrete.

References :

[1]. James R. Munkres, Topology, 2^{nd} ed.

[2]. Colin Adams and Robert Franzosa, Introduction to Topology : Pure and Applied.

[3]. R. Engelking, General Topology.

— o —