## Paracompactness :

**Def$^n$**: A family $\{A_i\}_{i\in I}$ of subsets of a top. space $X$ is said to be :

(i) locally finite if $\forall x \in X$ $\exists$ open subset $U$ of $X$ with $x \in U$ such that the set $\{i \in I \mid U \cap A_i \neq \phi\}$ is finite.

(ii) locally discrete if every $x \in X$ has an open nbd. which intersects with atmost one member of $\{A_i\}_{i\in I}$.

**Remark**: In a top. space $X$, every discrete family & finite family of sets is locally finite.

**Def$^n$**: (i) A family $\{A_i\}_{i\in I}$ of subsets of a top. space $X$ is said to be a cover of $X$ if $X = \bigcup_{i\in I} A_i$. Moreover, it is said to be an open (a closed) cover of $X$ if all sets $A_i$'s are open (closed). A subfamily $\{A_j\}_{j\in J\subseteq I}$ of cover $\{A_i\}_{i\in I}$ is said to a subcover of it if it also covers $X$, i.e., $X = \bigcup_{j\in J} A_j$.

(ii) A cover $\mathcal{B} = \{B_k\}_{k\in K}$ of $X$ is said to be a refinement of another cover $\mathcal{A} = \{A_i\}_{i\in I}$ of $X$ if $\forall k \in K$ $\exists$ an $i \in I$ such that $B_k \subseteq A_i$. In this situation, we also say that $\mathcal{B}$ refines $\mathcal{A}$.

$\underline{\text{Def}^n}$: (i) A top. space $X$ is said to be <u>paracompact</u> if every open cover of $X$ has a locally finite open refinement.

(ii) A top. space $X$ is said to be <u>compact</u> if every open cover of $X$ has a finite subcover.

<u>Remark</u> (i) It can be observed that every discrete space is paracompact - the cover consisting of all one-point sets is open and locally finite and refines any other cover of the space, whereas a discrete space is not compact, in general (in fact, when $X$ is infinite).

(ii) In the above definition of paracompactness the term "refinement" cannot be replaced by the term "subcover". As the open cover $\{\mathbb{N} \cap [1, i]\}_{i=1}^{\infty} = \{\{1\}, \{1,2\}, \{1,2,3\}, \cdots\}$ of the discrete space of natural numbers $\mathbb{N}$ has no locally finite subcover while this space $\mathbb{N}$ has a locally finite open refinement consisting of <u>all</u> one-point sets which refines every cover of the space.

Theorem: Every compact space is paracompact.

Proof: Left as an easy exercise #

**Definition:** $\mathcal{B}$ is said to be countably locally finite (or $\sigma$-locally finite) family of subsets of a top. space $X$ if $\mathcal{B}$ can be written as a countable union of $\mathcal{B}_n$, where each $\mathcal{B}_n$ is locally finite. In a similar manner, countably locally discrete family of subsets may be defined.

**Lemma:** In a regular space $X$, the following conditions are equi.

i) Every open covering of $X$ has an open refinement which is countably locally finite.

ii) Every open covering of $X$ has a locally finite refinement.

iii) Every open covering of $X$ has a locally finite closed refinement.

iv) Every open covering of $X$ has a locally finite open refinement.

**Pf:** i) $\Rightarrow$ ii). Let $\mathcal{A}$ be an open covering of $X$. Let $\mathcal{B}$ be a countably locally finite open refinement of $\mathcal{A}$. Let $\mathcal{B} = \cup \mathcal{B}_n$, where each $\mathcal{B}_n$ is locally finite. Now given any +ve integer $m$, let $V_m = \bigcup_{G \in \mathcal{B}_m} G$. Then each $V_m$ is open in $X$. Again, $\forall n \in \mathbb{Z}^+$ and $\forall G \in \mathcal{B}_n$, we define $S_n(G) = G - \bigcup_{m<n} V_m$.

(Here, we note that $S_n(G)$ is neither open nor closed, in general)

Let $\mathcal{C}_n = \{S_n(G) \mid G \in \mathcal{B}_n\}$. As $S_n(G) \subseteq G$, $\forall G \in \mathcal{B}_n$, so $\mathcal{C}_n$ is a refinement of $\mathcal{B}_n$. Take $\mathcal{C} = \cup \mathcal{C}_n$. Then we claim that $\mathcal{C}$ is the required locally finite refinement of $\mathcal{A}$. For this, let $x \in X$. Since $\mathcal{B} = \cup \mathcal{B}_n$ is a cover of $X$, so there are some $\mathcal{B}_n$'s which contains $x$. Let $k$ be the smallest +ve integer such that $x$ lies in some element of $\mathcal{B}_k$. Let $U \in \mathcal{B}_k$ such that $x \in U$. Note that $x$ does not lie in any element of $\mathcal{B}_i$ for $i < k$, so $x \in S_k(U) \in \mathcal{C}_k \in \mathcal{C}$.

Thus $\mathcal{C}$ covers $X$. Next, since each collection $\mathcal{B}_n$ is locally finite, we can get for each $n = 1, 2, \cdots, k$ an open set $W_n$ with $x \in W_n$ such that $W_n$ intersects with only finitely many elements of $\mathcal{B}_n$. Now if $W_n \cap S_n(V) \neq \phi$, then clearly $W_n \cap V \neq \phi$, where $V \in \mathcal{B}_n$ ($\because S_n(V) \subseteq V$). Therefore, $W_n$ intersects with only finitely many elements of $\mathcal{C}_n$. As $U \in \mathcal{B}_k$, so $U$ does not intersect with any element of $\mathcal{C}_n$ for $n > k$. Consequently, the open set $W_1 \cap \cdots \cap W_k \cap U$ contains $x$ and it intersects with only finitely many elements of $\mathcal{C}$.

ii) $\Rightarrow$ iii). Let $\mathcal{A}$ be an open cover of $X$. Let $\mathcal{B}$ be the collection of all open sets $U$ of $X$ such that $\overline{U}$ is contained in some element of $\mathcal{A}$. $\mathcal{B}$ covers $X$, being $X$ regular. Now using (ii), we can find a refinement $\mathcal{C}$ of $\mathcal{B}$ that covers $X$ and is locally finite. Take $\mathcal{D} = \{\overline{C} \mid C \in \mathcal{C}\}$. Then $\mathcal{D}$ turns out to be a locally finite refinement of $\mathcal{A}$ (Exercise).

iii) $\Rightarrow$ iv). Leave as an exercise. (For details, see [1]).

iv) ⇒ i). Obvious. #

Theorem: Every metrizable space is paracompact.

Pf: (Outline only). Since in a metrizable space every open cover has a countably locally finite open refinement, so by the above lemma it is paracompact. #

Theorem: Every regular Lindelöf space is paracompact.

Pf: Let the space $X$ be regular and Lindelöf. Then given any open cover $\mathcal{A}$ of $X$, it has a countable subcover. Clearly, this subcover is countably locally finite. Hence, by the above lemma $X$ is paracompact. #

References:


[1] James R. Munkres, Topology, $2^{nd}$ ed.

[2] R. Engelking, General Topology.


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