Lecture-XI Programme: M. Sc. Physics

Liénard - Wiechert Fields due to a Point Charge

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The Fields of a Moving Point Charge

❖ In this lecture we will study about Liénard-Wiechert fields due to a moving point charge.

❖ To find the expressions for the fields (the electric and magnetic fields) of a point charge in arbitrary motion, we will use expressions of the Liénard-Wiechert potentials from the previous lecture X:

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{qc}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})}, \quad \text{-------------} \quad [1]$$

and

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mathbf{v}}{c^2} V(\mathbf{r}, t), \quad \text{-------------} \quad [2]$$

❖ As we know that relations for E and B:

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}. \qquad \text{------------- [3]}$$

❖ The differentiation is complicated, however, as

$$\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{w}(t_r) \quad \text{and} \quad \mathbf{v} = \dot{\mathbf{w}}(t_r) \qquad \text{------------- [4]}$$

are both determined at the retarded time, and t_r described implicitly by the equation

$$|\mathbf{r} - \mathbf{w}(t_r)| = c(t - t_r) \qquad \text{------------- [5]}$$

is a function of r and *t*.

❖ Let's start with the gradient of *V*:

$$\nabla V = \frac{qc}{4\pi\epsilon_0} \frac{-1}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})^2} \nabla(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v}). \qquad \text{-------------} \quad [6]$$

Since $\mathscr{r} = c(t - t_r)$,

$$\nabla \mathscr{r} = -c\nabla t_r \qquad \text{-------------} \quad [7]$$

As for the second term, product rule provides

$$\nabla(\boldsymbol{\mathscr{r}} \cdot v) = (\boldsymbol{\mathscr{r}} \cdot \nabla)v + (v \cdot \nabla)\boldsymbol{\mathscr{r}} + \boldsymbol{\mathscr{r}} \times (\nabla \times v) + v \times (\nabla \times \boldsymbol{\mathscr{r}}) \qquad \text{-------------} \quad [8]$$

❖ **Calculating these terms one at a moment:**

$$(\boldsymbol{\mathscr{r}} \cdot \nabla)\mathbf{v} = \left(\mathscr{r}_x \frac{\partial}{\partial x} + \mathscr{r}_y \frac{\partial}{\partial y} + \mathscr{r}_z \frac{\partial}{\partial z}\right)\mathbf{v}(t_r)$$

$$= \mathscr{r}_x \frac{d\mathbf{v}}{dt_r}\frac{\partial t_r}{\partial x} + \mathscr{r}_y \frac{d\mathbf{v}}{dt_r}\frac{\partial t_r}{\partial y} + \mathscr{r}_z \frac{d\mathbf{v}}{dt_r}\frac{\partial t_r}{\partial z}$$

$$= \mathbf{a}(\boldsymbol{\mathscr{r}} \cdot \nabla t_r), \qquad \text{------------- [9]}$$

Where $\mathbf{a} \equiv \dot{\mathbf{v}}$ is the *acceleration* of the particle at the retarded time. Now

$$(\mathbf{v} \cdot \nabla)\, \boldsymbol{\mathscr{r}} = (\mathbf{v} \cdot \nabla)\, \mathbf{r} - (\mathbf{v} \cdot \nabla)\mathbf{w} \qquad \text{------------- [10]}$$

and

$$(\mathbf{v}\cdot\nabla)\mathbf{r} = \left(v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z}\right)(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}})$$
$$= v_x\,\hat{\mathbf{x}} + v_y\,\hat{\mathbf{y}} + v_z\,\hat{\mathbf{z}} = \mathbf{v},$$

------------- [11]

while $(\mathbf{v}\cdot\nabla)\mathbf{w} = \mathbf{v}(\mathbf{v}\cdot\nabla t_r)$

(similar reasoning as Eq. 9). Moving on to the third term in Eq. 8,

$$\nabla\times\mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

$$\nabla \times \mathbf{v} = \left(\frac{dv_z}{dt_r} \frac{\partial t_r}{\partial y} - \frac{dv_y}{dt_r} \frac{\partial t_r}{\partial z} \right) \hat{\mathbf{x}} + \left(\frac{dv_x}{dt_r} \frac{\partial t_r}{\partial z} - \frac{dv_z}{dt_r} \frac{\partial t_r}{\partial x} \right) \hat{\mathbf{y}} + \left(\frac{dv_y}{dt_r} \frac{\partial t_r}{\partial x} - \frac{dv_x}{dt_r} \frac{\partial t_r}{\partial y} \right) \hat{\mathbf{z}}$$

$$\nabla \times \mathbf{v} = - \mathbf{a} \times \nabla t_r \qquad \text{-------------} \quad [12]$$

Finally,

$$\nabla \times \boldsymbol{\mathscr{r}} = \nabla \times \mathbf{r} - \nabla \times \mathbf{w} \qquad \text{-------------} \quad [13]$$

although $\nabla \times \mathbf{r} = 0$, whilst, by the similar argument as Eq. 12,

$$\nabla \times w = - v \times \nabla t_r \qquad \text{-------------} \quad [14]$$

❖ **Substituting all this back into Eq. 8, and utilizing the "BAC-CAB" rule to short the triple cross products,**

$$\nabla(\boldsymbol{\mathscr{r}} \cdot \mathbf{v}) = \mathbf{a}(\boldsymbol{\mathscr{r}} \cdot \nabla t_r) + \mathbf{v} - \mathbf{v}(\mathbf{v} \cdot \nabla t_r) - \boldsymbol{\mathscr{r}} \times (\mathbf{a} \times \nabla t_r) + \mathbf{v} \times (\mathbf{v} \times \nabla t_r)$$

$$\nabla(\boldsymbol{\mathscr{r}} \cdot \mathbf{v}) = \mathbf{v} + (\boldsymbol{\mathscr{r}} \cdot \mathbf{a} - v^2)\, \nabla t_r$$ ------------- [15]

assembling Eqs. 7 and 15, we have

$$\nabla V = \frac{qc}{4\pi\epsilon_0} \frac{1}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})^2} \left[\mathbf{v} + (c^2 - v^2 + \boldsymbol{\mathscr{r}} \cdot \mathbf{a})\nabla t_r\right].$$ ------------- [16]

❖ To finalize this formulation, we require to know ∇t_r. This can be established by taking the gradient from equation (Eq. 5) which we have already done in Eq. 7—and expanding $\nabla \mathscr{r}$:

$$-c\nabla t_r = \nabla \mathscr{r} = \nabla\sqrt{\boldsymbol{\mathscr{r}}\cdot\boldsymbol{\mathscr{r}}} = \frac{1}{2\sqrt{\boldsymbol{\mathscr{r}}\cdot\boldsymbol{\mathscr{r}}}}\nabla(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{\mathscr{r}})$$

$$-c\,\nabla t_r = \frac{1}{\mathscr{r}}\left[(\boldsymbol{\mathscr{r}}\cdot\nabla)\boldsymbol{\mathscr{r}} + \boldsymbol{\mathscr{r}}\times(\nabla\times\boldsymbol{\mathscr{r}})\right].$$

-------------- [17]

But

$$(\boldsymbol{\tau} \cdot \nabla)\, \boldsymbol{\tau} = \boldsymbol{\tau} - \mathrm{v}(\boldsymbol{\tau} \cdot \nabla\, t_r)$$

-------------- [18]

(similar thought as Eq. 10), while (from Eqs. 13 and 14)

$$\nabla \times \boldsymbol{\tau} = (v \times \nabla t_r)$$

-------------- [19]

Thus

$$-c\, \nabla t_r = 1/\boldsymbol{\tau}[\boldsymbol{\tau} - \mathrm{v}(\boldsymbol{\tau} \cdot \nabla tr) + \boldsymbol{\tau} \times (\mathrm{v} \times \nabla t_r)] = 1/\boldsymbol{\tau}\, [\boldsymbol{\tau} - (\boldsymbol{\tau} \cdot \mathrm{v})\, \nabla t_r]$$

-------------- [20]

and thus

$$\nabla t_r = \frac{-\boldsymbol{\mathscr{r}}}{\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v}}. \qquad \text{-------------- [21]}$$

Incorporating this result into Eq. 16, we finish that

$$\nabla V = \frac{1}{4\pi\epsilon_0} \frac{qc}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})^3} \left[(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})\mathbf{v} - (c^2 - v^2 + \boldsymbol{\mathscr{r}} \cdot \mathbf{a})\boldsymbol{\mathscr{r}} \right]. \qquad \text{---------- [22]}$$

A similar calculation,

$$\frac{\partial \mathbf{A}}{\partial t} = \frac{1}{4\pi\epsilon_0} \frac{qc}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})^3} \left[(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})(-\mathbf{v} + \mathscr{r}\mathbf{a}/c) + \frac{\mathscr{r}}{c}(c^2 - v^2 + \boldsymbol{\mathscr{r}} \cdot \mathbf{a})\mathbf{v} \right]. \qquad \text{---------- [23]}$$

merging these outcomes, and setting up the vector

$$\mathbf{u} \equiv c\hat{\boldsymbol{\mathscr{r}}} - \mathbf{v},$$

We get

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}} \cdot \mathbf{u})^3} \left[(c^2 - v^2)\mathbf{u} + \boldsymbol{\mathscr{r}} \times (\mathbf{u} \times \mathbf{a}) \right].$$ ---------- [24]

Meanwhile

$$\nabla \times \mathbf{A} = \frac{1}{c^2} \nabla \times (V\mathbf{v}) = \frac{1}{c^2} [V(\nabla \times \mathbf{v}) - \mathbf{v} \times (\nabla V)].$$ ---------- [25]

❖ We have already determined $\nabla \times \mathbf{v}$ (Eq. 12) and ∇V (Eq. 22). Substituting these jointly,

$$\nabla \times \mathbf{A} = -\frac{1}{c}\frac{q}{4\pi\epsilon_0}\frac{1}{(\mathbf{u}\cdot\boldsymbol{\mathscr{r}})^3}\boldsymbol{\mathscr{r}} \times \left[(c^2 - v^2)\mathbf{v} + (\boldsymbol{\mathscr{r}}\cdot\mathbf{a})\mathbf{v} + (\boldsymbol{\mathscr{r}}\cdot\mathbf{u})\mathbf{a}\right].$$

---------- [26]

❖ The quantity in brackets is noticeably similar to to the one in Eq. 24, which can be marked, using the BAC-CAB rule, as $[(c^2 - v^2)\mathbf{u} + (\mathscr{r}\cdot\mathbf{a})\mathbf{u} - (\mathscr{r}\cdot\mathbf{u})\mathbf{a}]$;

❖The key difference is that we have v's in its place of u's in the initial two terms. In reality, as it is all crossed into $\mathscr{r}$ anyway, we can with impunity alter these v's into −u's; the additional term proportional to $\mathscr{r}$ vanishes in the cross product.

❖ It follows that

$$\mathbf{B}(\mathbf{r}, t) = \frac{1}{c}\hat{\boldsymbol{\mathscr{r}}} \times \mathbf{E}(\mathbf{r}, t).$$

---------- [27]

❖ It is clear that *the magnetic field of a point charge is always perpendicular to the electric field, and to the vector from the retarded point.*

❖ The first term in equation (24), E (the one connecting *($c^2 - v^2$)*u falls off as the *inverse square* of the distance from the particle.

❖ If the velocity and acceleration are both zero, then the equation (24) turns into the old electrostatic result

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}.$$

---------- [28]

❖ For this basis, the first term in E is sometimes known as the well familiar ***Coulomb field***. (since it does not depend on the acceleration, it is also called the velocity field.)

❖The second term (the one linking $\mathbf{\mathscr{r}} \times (\mathbf{u} \times \mathbf{a})$) falls off as the inverse *first* power of $\mathscr{r}$ and is thus leading at large distances. This term is responsible for **electromagnetic radiation**; hence, it is known as the **radiation field**—or, since it is proportional to *a*, the acceleration field.

Numerical problems

1. Calculate the electric and magnetic fields of a point charge moving with constant velocity.

2. Suppose a point charge q is constrained to move along the x axis. Show that the fields at points on the axis to the right of the charge are given by.

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{1}{\eta^2}\left(\frac{c+v}{c-v}\right)\hat{\mathbf{x}}, \quad \mathbf{B} = \mathbf{0}.$$

(Do not assume *v* is constant!) What are the fields on the axis to the *left* of the charge?

3. For a point charge moving at constant velocity, calculate the flux integral $\oint \mathbf{E} \cdot d\mathbf{a}$ (using Eq. 10.75), over the surface of a sphere centered at the present location of the charge.

References:

1. *Introduction to Electrodynamics, David J. Griffiths*

2. *Elements of Electromagnetics, 2nd edition by M N O Sadiku*

3. *Engineering Electromagnetics by W H Hayt and J A Buck.*

4. *Elements of Electromagnetic Theory & Electrodynamics, Satya Prakash*

- **For any query/ problem contact me on whatsapp group or mail on me**

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- **Next *** we will discuss electric dipole radiation and numerical problems based on radiation topic.**

Thank you