Lecture Notes
Part 21

PHYS 3014: Statistical Mechanics

Relativistic Fermi Gas

Dr. Ajai Kumar Gupta
Professor
Department of Physics
Mahatma Gandhi Central University
Motihari-845401, Bihar
E-mail: akgupta@mgcub.ac.in

Programme: B. Sc. Physics
Semester: VI

Relativistic Fermi Gas at T = 0 K.

Let Z be the grand partition function of the relativistic Fermi gas. Then

$$\ln Z = \sum_{k} \ln(1 + z e^{-\beta \epsilon_k}) \quad \text{---} \quad (1)$$

The single particle energies ϵ_k in momentum space k is relativistic and can be written as.

$$\epsilon = mc^2 \left[\sqrt{1 + \left(\frac{p}{mc}\right)^2} - 1 \right] \quad \text{---} \quad (2)$$

when $\left(\frac{p}{mc}\right) \to 0$, ϵ is negligible in comparison to mc^2 rest mass energy representing non-relativistic case and when $\left(\frac{p}{mc}\right) \to \infty$, $\frac{\epsilon}{mc^2}$ also tends to infinity representing completely relativistic case.

The average no. of particle in momentum space k

is

$$\bar{n}_k = \frac{1}{z^{-1} e^{\beta \epsilon_k} + 1} \quad \text{---} \; (3)$$

where $z = e^{\mu/KT}$ is the fugacity of the system and is determined from particle number of the system.

For large volume V

$$\ln Z = g \frac{4\pi V}{h^3} \int_0^\infty p^2 dp \ln(1 + z e^{-\beta \epsilon}) \quad \text{---} \; (4)$$

and

$$N(T, V, z) = g \frac{4\pi V}{h^3} \int_0^\infty p^2 dp \frac{1}{z^{-1} e^{\beta \epsilon} + 1} \quad \text{---} \; (5)$$

Integrating equ^n (4) by parts,

$$\ln Z = g \frac{4\pi V}{h^3} \cdot \frac{\beta}{3} \int_0^\infty p^3 dp \frac{d\epsilon}{dp} \frac{1}{z^{-1} e^{\beta \epsilon} + 1} \quad \text{--} \; (6)$$

At $T = 0$, the mean occupation number $\bar{n}_K$ has the form of step function $(\bar{n}_K)_{FD} = \Theta(\epsilon_F - \epsilon)$ i.e $\bar{n}_K$ is 1 when $\epsilon < \epsilon_F$ and is 0 when $\epsilon > \epsilon_F$.

$\therefore$ eqn (6) becomes

$$\ln z = g \frac{4\pi V}{h^3} \frac{\beta}{3} \int_0^{p_f} p^3 dp \frac{d\epsilon}{dp} \quad \text{--- (7)}$$

where p_f is the momentum corresponding to the fermi energy ϵ_F.

Equation (5) becomes

$$N(T, V, z) = g \frac{4\pi V}{h^3} \cdot \frac{p_f^3}{3} \quad \text{--- (8)}$$

$$\therefore \quad p_f = \left(\frac{3}{4\pi} \frac{N h^3}{V g} \right)^{1/3} \quad \text{--- (9)}$$

at 0K, $\mu = \epsilon_F(0)$

$$z = e^{\frac{\mu}{KT}} = e^{\frac{\epsilon_F(0)}{KT}}$$

$\epsilon_F = \frac{p_f^2}{2m}$ is the Fermi energy at 0K.

Differentiating equ^n (9)

$$\frac{d\epsilon}{dp} = \frac{1}{2} mc^2 \frac{1 \cdot \frac{2p}{m^2c^2}}{\sqrt{1+\left(\frac{p}{mc}\right)^2}}$$

$$= c \frac{\left(\frac{p}{mc}\right)}{\sqrt{1+\left(\frac{p}{mc}\right)}} \quad \text{---} \quad (10)$$

By equ^n (7) and (10) we get

$$\ln Z = \frac{PV}{KT} = \frac{4\pi g}{3h^3} \cdot V\beta \int_0^{p_f} mc^2 \cdot \frac{\left(\frac{p}{mc}\right)^2 p^2 dp}{\left[1+\left(\frac{p}{mc}\right)^2\right]^{1/2}} \quad \text{--} \quad (11)$$

or, Pressure in the system

$$P = \frac{4\pi g}{3h^3} \int_0^{p_f} mc^2 \frac{\left(\frac{p}{mc}\right)^2 p^2 dp}{\left[1+\left(\frac{p}{mc}\right)^2\right]^{1/2}} \quad \text{---} \quad (12)$$

The energy of the system.

$$U = \sum_k \epsilon_k \bar{n}_k \quad \text{---} \quad (13)$$

For large volume V,

$$U = \frac{4\pi g V}{h^3} \int_0^{p_f} p^2 dp \; mc^2 \left[\left\{ 1 + \left(\frac{p}{mc}\right)^2 \right\}^{1/2} - 1 \right] \quad \text{at } T = 0\,K \qquad \text{--- (14)}$$

Substitute $\frac{p}{mc} = \sinh x$ in eqn (12) and (14)

$\therefore \quad \epsilon = mc^2(\cosh x - 1)$ and $\frac{d\epsilon}{dp} = c \tanh x$

we get

$$P = \frac{4\pi g m^4 c^5}{3h^3} \int_0^{x_f} \sinh^4 x \, dx \qquad \text{--- (15)}$$

$p_f = mc \sinh x_f$

and

$$U = \frac{4\pi g V m^4 c^5}{h^3} \int_0^{x_f} (\cosh x - 1) \sinh^2 x \cdot \cosh x \, dx \qquad \text{--- (16)}$$

For $y = \sinh x = \frac{p}{mc}$ and $y_f = \sinh x_f = \frac{p_f}{mc}$

new notation, integrals in

Eqn (15) and (16) can be written as

$$\int_0^{x_f} \sinh^4 x \, dx = \frac{1}{8} A(y_f)$$

and
$$\int_0^{x_f} (\cosh x - 1) \sinh^2 x \cosh x \, dx = \frac{1}{24} B(y_f)$$

where $A(y_f)$ and $B(y_f)$ are defined by

$$A(y) = \sqrt{1+y^2}\,(2y^3 - 3y) + 3 \,\mathrm{ArcSinh}\, y$$

and
$$B(y) = 8y^3\left(\sqrt{1+y^2} - 1\right) - A(y)$$

In the limiting case when $y << 1$

$$A(y) \simeq \frac{8}{5} y^5 - \frac{4}{7} y^7 + \cdots$$

$$B(y) \simeq \frac{12}{5} y^5 - \frac{3}{7} y^7 + \cdots$$

when $y >> 1$
$$A(y) \simeq 2y^4 - 2y^2 + 3\ln(2y) - \frac{7}{4} + \frac{5}{4} y^{-2} + \cdots$$

$$B(y) \simeq 6y^4 - 8y^3 + 6y^2 - 3\ln(2y) + \frac{3}{4} - \frac{3}{4} y^{-2} + \cdots$$

Therefore equ^n (15) and (16) becomes

$$P = \frac{g\pi m^4 c^5}{6h^3} A(y_f) \quad \text{- - - (16)}$$

$$U = \frac{g\pi V m^4 c^5}{6h^3} B(y_f) \quad \text{- - (17)}$$

For non-relativistic case $y_f << 1$

$$\therefore \quad P = \frac{g\pi m^4 c^5}{6h^3} \cdot \frac{8}{5} y_f^5$$

$$= \frac{g\pi m^4 c^5}{6h^3} \cdot \frac{8}{5} \cdot \left(\frac{p_f}{mc}\right)^5$$

$$= \frac{2}{3}\left(\frac{2\pi g}{5mh^3}\right) p_f^5$$

and $$U = \frac{g\pi V m^4 c^5}{6h^3} \cdot \frac{12}{5} y_f^5 = \frac{2}{5} \frac{g\pi V m^4 c^5}{h^3} \frac{p_f^5}{(mc)^5}$$

$$= \frac{2\pi g V}{5mh^3} p_f^5$$

$\therefore$ $\boxed{P = \frac{2}{3}\frac{U}{V}}$ is like ideal Fermi gas.

For ultrarelativistic case $y_f >> 1$

$$P = \frac{g\pi m^4 c^5}{6h^3} \cdot 2 (y_f)^4$$

$$= \frac{1}{3} g\pi \frac{m^4c^5}{h^3} \left(\frac{p_f}{mc}\right)^4$$

$$= \frac{1}{3}\left(\frac{g\pi c}{h^3} p_f^4\right)$$

and

$$U = \frac{g\pi V m^4 c^5}{6h^3} 6 (y_f)^4$$

$$= \frac{g\pi V c}{h^3} p_f^4$$

$\therefore$ $\boxed{P = \frac{1}{3}\frac{U}{V}}$

References:

- Statistical Mechanics by R. K. Pathria
- Statistical Mechanics by B. K. Agarwal and M. Eisner
- An Introductory Course of Statistical Mechanics by P. B. Pal
- Elementary Statistical Physics by C. Kittel
- Fundamentals of Statistical and Thermal Physics by F. Reif
- Statistical and Thermal Physics by R. S. Gambhir and S. Lokanathan

Thank You

For any questions/doubts/suggestions and submission of assignments

Write at E-mail: akgupta@mgcub.ac.in