PHYS 3014: Statistical Mechanics

Lecture Notes
Part 22

White Dwarf Stars & Chandrasekhar Mass Limit

Dr. Ajai Kumar Gupta

Professor
Department of Physics
Mahatma Gandhi Central University
Motihari-845401, Bihar
E-mail: akgupta@mgcub.ac.in

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White Dwarf Stars; Chandrasekhar Mass Limit

White dwarf stars are old stars. They are at end phase of their life. They have small diameters so they are extremely dense. Their colour is white and luminosity is very less in comparison to other stars.

Normally a star with white colour is expected to be brilliant star whereas star with red colour is expected to be dull star. White dwarfs are exceptions as they are relatively old, most of their hydrogen content are used so thermonuclear reactions are occuring at lower rate giving them to faint or less bright colour.

Most of the material content of Dwarf stars are helium. The brightness displayed by dwarf stars are due to gravitational energy.

released during slow contraction of these stars.
(Mechanism proposed by Kelvin 1861).

White dwarf star

Mass $M \simeq 10^{30}$ kg most of helium

mass density $\rho \simeq 10^7\ g/cm^3$

central temperature $T \simeq 10^7$ K

Mean thermal energy per particle $\simeq 10^3$ eV

The mean thermal energy is much greater than the energy required to ionized helium atom (~55 eV) so all the helium atoms in white dwarf stars are completely ionized.

The microscopic constituents of white dwarf stars can be considered as N electrons and $\frac{N}{2}$ ionized helium atom.

mass of star

$$M = N m_e + \frac{N}{2} \times 4 m_p$$

$$= N(m_e + 2m_p)$$

$$\therefore M \simeq 2 m_p N \quad \because m_e << m_p \quad \text{--- (1)}$$

m_e = mass of electron

m_p = mass of proton

$4m_p$ = mass of helium nucleus.

Particle density of electrons in the star

$$n = \frac{N}{V} \simeq \frac{M/2m_p}{M/\rho} = \frac{\rho}{2m_p} \simeq 3\times10^{36} \frac{\text{electrons}}{m^3}$$

The fermi momentum of electron gas corresponding to this density

$$p_f = \left(\frac{3n}{4\pi g}\right)^{1/3} h \simeq 5\times10^{-22} \ kg\ m/sec \quad \because g=2 \text{ for electrons}$$

$$\simeq 0.9 \frac{MeV}{c}$$

The fermi energy corresponding to this momentum

$$E_f \simeq 0.77 MeV$$

The rest energy of electron $\simeq$ 0.5 MeV. Therefore relativistic effects become important.

$$\text{Fermi temperature } T_F = \frac{\epsilon_F}{K}$$

$$\simeq \frac{0.77 \times 10^{6} \times 1.6 \times 10^{-19}}{1.38 \times 10^{-23}}$$

$$\simeq 1.6 \times .77 \times \frac{10^{10}}{1.38}$$

$$\simeq 0.89 \times 10^{10} \text{ K}$$

$\therefore$ for electron gas in the star

$$\frac{T}{T_F} \simeq \frac{10^{7}}{10^{10}} = 10^{-3} \ll 1$$

Therefore, electron gas in the dwarf star is in a state of almost complete degeneracy. And the dynamics of electrons in a dwarf star is relativistic.

Helium nuclei do not contribute significantly to the system. The pressure exerted by electron gas at absolute zero is very high in comparison to helium nuclei. Therefore we can assume a dwarf star as sphere consisting of highly dense electron gas. The electron gas is uniformly distributed over the body of the star. The enormous pressure exerted by electron gas in dwarf star is counterbalanced by the binding due to the gravitational attraction which is entirely due to the helium nuclei in the star.

Now near to $T=0$, for fermi gas composed of N relativistic electrons, we have

$$N = g \frac{4\pi V}{h^3} \int_0^{p_f} p^2 dp = \frac{8\pi V}{3h^3} p_f^3 \qquad \because g=2 \quad \text{--- (1)}$$

$$\therefore \quad p_f = \left(\frac{3N}{8\pi V}\right)^{1/3} h \quad \text{--- (2)}$$

For relativistic electrons, energy

$$\epsilon = mc^2\left[\left\{1+\left(\frac{p}{mc}\right)^2\right\}^{1/2}-1\right] \quad \text{--- (3)}$$

m is rest mass of electrons.

The speed of electrons

$$v = \frac{d\epsilon}{dp} = \frac{\frac{p}{m}}{\left\{1+\left(\frac{p}{mc}\right)^2\right\}^{1/2}} \quad \text{--- (4)}$$

∴ The pressure of electron gas

$$P_0 = \frac{1}{3}\frac{N}{V}\langle pv\rangle_0 = \frac{1}{3}\frac{N}{V}\left\langle p\frac{d\epsilon}{dp}\right\rangle_0$$

$$= \frac{8\pi}{3h^3}\int_0^{p_f} \frac{p^2/m}{\left[1+\left(\frac{p}{mc}\right)^2\right]^{1/2}}\, p^2\,dp \quad \text{--- (5)}$$

Let $\sinh x = \frac{p}{mc} = y \Rightarrow y_f = \frac{p_f}{mc} = \sinh x_f$

∴ equn (1) becomes

$$N = \frac{8\pi V m^3 c^3}{3h^3}\, y_f^3 \quad \text{--- (6)}$$

So $\quad R \propto M^{-1/3}$ --- (19)

(ii) when $\quad R \ll 10^8$ cm, i.e $\quad y_f \gg 1$

then

$$A(y_f) \simeq 2y_f^4 - 2y_f^2$$

$\therefore$ eqn. (16) becomes

$$2y_f^2\left[y_f^2 - 1\right] = 6\pi\alpha\left(\frac{\hbar/mc}{R}\right)^3\left(\frac{GM^2/R}{mc^2}\right)$$

or,

$$\left(\frac{9\pi M}{8m_p}\right)^{2/3}\left(\frac{\hbar/mc}{R}\right)^2\left[\left(\frac{9\pi M}{8m_p}\right)^{2/3}\left(\frac{\frac{\hbar}{mc}}{R}\right)^2 - 1\right] = 3\pi\alpha\left(\frac{\hbar/mc}{R}\right)^3\left(\frac{GM^2/R}{mc^2}\right)$$

or,

$$\left(\frac{9\pi M}{8m_p}\right)^{4/3}\left(\frac{\hbar/mc}{R}\right)^2 - \left(\frac{9\pi M}{8m_p}\right)^{2/3} = 3\pi\alpha\cdot\left(\hbar/mc\right)\cdot\frac{GM^2}{mc^2}\cdot\frac{1}{R^2}$$

or,

$$\left[\left(\frac{9\pi M}{8m_p}\right)^{4/3}\left(\frac{\hbar}{mc}\right)^2 - 3\pi\alpha\,\frac{\hbar}{mc}\cdot\frac{GM^2}{mc^2}\right]\frac{1}{R^2} = \left(\frac{9\pi M}{8m_p}\right)^{2/3}$$

The electron gas exerts a pressure $P_0(n)$ on the wall and any compression or expansion of gas will always accompanied with some work.

For adiabatic change in volume of gas, the change in energy of the gas is

$$dE_0 = -P_0(n)dv$$

or

$$dE_0 = -P_0(R)\cdot 4\pi R^2 dR \quad --- (9)$$

At the same time when sphere is enlarged, then its potential energy increases by

$$dE_g = \frac{dE_g(R)}{dR}dR$$

$$= \alpha \frac{GM^2}{R^2} dR \quad --- (10)$$

where $E_g(R) = -\alpha \frac{GM^2}{R}$ and α represents the correction due to inhomogenous density distribution.

At equilibrium, the net change in total energy ($E_0 + E_g$) must be zero for in infinitesimal change in size of the system at $T = 0K$.

$$dE_0 + dE_g = 0 \quad \text{---} \quad (11)$$

or,
$$-P_0(R)\, 4\pi R^2 dR + \alpha \frac{GM^2}{R^2} dR = 0$$

or,
$$P_0(R) = \frac{\alpha}{4\pi} \frac{GM^2}{R^4} \quad \text{---} \quad (12)$$

From equn (7)
$$P_0(R) = \frac{\pi m^4 c^5}{3h^3} A(y_f) \quad \text{---} \quad (13)$$

by (12) and (13)

$$\frac{\pi m^4 c^5}{3h^3} A(y_f) = \frac{\alpha}{4\pi} \frac{Gm^2}{R^4}$$

or,
$$A(y_f) = \frac{3\alpha h^3}{4\pi^2 m^4 c^5} \frac{GM^2}{R^4}. \quad \text{---} \quad (14)$$

and $\quad y_f = \frac{p_f}{mc} = \left(\frac{3N}{8\pi V}\right)^{1/3} \frac{h}{mc} \quad , \quad V = \frac{4\pi}{3} R^3$

$$y_f = \left(\frac{9N}{32\pi^2}\right)^{1/3} \frac{h/mc}{R}$$

$$= \left(\frac{9M}{64\pi^2 m_p}\right)^{1/3} \frac{h/mc}{R} \quad \text{---} (15) \qquad \because M = 2N m_p$$

$\therefore$ equn (14) becomes.

$$A\left[\left(\frac{9M}{64\pi^2 m_p}\right)^{1/3}\left(\frac{h/mc}{R}\right)\right] = 6\pi\alpha\left(\frac{\hbar/mc}{R}\right)^3 \frac{GM^2/R}{mc^2}$$

or,

$$A\left[\left(\frac{9\pi M}{8 m_p}\right)^{1/3}\left(\frac{\hbar/mc}{R}\right)\right] = 6\pi\alpha\left(\frac{\hbar/mc}{R}\right)^3 \cdot \left(\frac{GM^2/R}{mc^2}\right) \quad \text{---} (16)$$

Equn (16) represents mass - radius relationship for dwarf stars. Here mass of star is measured in units of nucleon mass m_p, radius of star is measured in units of Compton wavelength of electrons $\frac{\hbar c}{mc^2}$ and gravitational energy $\frac{GM^2}{R}$ in the units of electron rest energy. The equation (16) represents a relationship among Quantum mechanics, special relativity and classical gravitational theory.

The quantity y_f is of the order of unity when $R \sim 10^8$ cm. $\left\{ \because M \sim 10^{33} g,\ m_p \sim 10^{-24} g,\ \frac{\hbar}{mc} \sim 10^{-11} cm \right\}$

Therefore

(i) When $R >> 10^8$ cm i.e $y_f << 1$

$$\therefore \quad A(y_f) \simeq \frac{8}{5} y_f^5 \quad \text{---} \ (17)$$

$\therefore$ equn (16) becomes

$$\frac{8}{5}\left(\frac{9\pi M}{8 m_p}\right)^{\frac{5}{3}} \left(\frac{\hbar/mc}{R}\right)^5 \simeq 6\pi\alpha \left(\frac{\frac{\hbar}{mc}}{R}\right)^3 \left(\frac{GM^2/R}{mc^2}\right)$$

$$\Rightarrow \quad R \simeq \frac{3}{40\alpha} (9\pi)^{2/3} \frac{\hbar^2}{G m m_p^{5/3}} M^{-1/3} \quad \text{---} \ (18)$$

So $\quad R \propto M^{-1/3} \quad - - - (19)$

(ii) when $R \ll 10^8$ cm, i.e $y_f \gg 1$

then

$$A(y_f) \simeq 2y_f^4 - 2y_f^2$$

$\therefore$ equn (16) becomes

$$2y_f^2\left[y_f^2 - 1\right] = 6\pi\alpha\left(\frac{\hbar/mc}{R}\right)^3\left(\frac{GM^2/R}{mc^2}\right)$$

or,

$$\left(\frac{9\pi M}{8m_p}\right)^{2/3}\left(\frac{\hbar/mc}{R}\right)^2\left[\left(\frac{9\pi M}{8m_p}\right)^{2/3}\left(\frac{\frac{\hbar}{mc}}{R}\right)^2 - 1\right] = 3\pi\alpha\left(\frac{\hbar/mc}{R}\right)^3\left(\frac{GM^2/R}{mc^2}\right)$$

or,

$$\left(\frac{9\pi M}{8m_p}\right)^{4/3}\left(\frac{\hbar/mc}{R}\right)^2 - \left(\frac{9\pi M}{8m_p}\right)^{2/3} = 3\pi\alpha\,(\hbar/mc)\,\frac{GM^2}{mc^2}\cdot\frac{1}{R^2}$$

or,

$$\left[\left(\frac{9\pi M}{8m_p}\right)^{4/3}\left(\frac{\hbar}{mc}\right)^2 - 3\pi\alpha\,\frac{\hbar}{mc}\,\frac{GM^2}{mc^2}\right]\frac{1}{R^2} = \left(\frac{9\pi M}{8m_p}\right)^{2/3}$$

or, $$R^2 = \left(\frac{8m_p}{9\pi M}\right)^{2/3}\left[\left(\frac{9\pi M}{8m_p}\right)^{4/3}\left(\frac{\hbar}{mc}\right)^2 - 3\pi\alpha\left(\frac{\hbar}{mc}\right)\cdot\left(\frac{GM^2}{mc^2}\right)\right]$$

$$= \left(\frac{9\pi M}{8m_p}\right)^{2/3}\left(\frac{\hbar}{mc}\right)^2\left[1 - 3\pi\alpha\cdot\frac{\hbar}{mc}\cdot\left(\frac{mc}{\hbar}\right)^2\cdot\frac{GM^2}{mc^2}\left(\frac{8m_p}{9\pi M}\right)^{4/3}\right]$$

or, $$R \simeq \left(\frac{9\pi M}{8m_p}\right)^{1/3}\left(\frac{\hbar}{mc}\right)\left[1 - 3\pi\alpha\,\frac{mc}{\hbar}\cdot\frac{G}{mc^2}\left(\frac{8m_p}{9\pi}\right)^{4/3}\cdot M^{2/3}\right]^{1/2}$$

or, $$R \simeq \frac{(9\pi)^{1/3}}{2}\left(\frac{\hbar}{mc}\right)\left(\frac{M}{m_p}\right)^{1/3}\left[1 - \left(\frac{M}{M_0}\right)^{2/3}\right]^{1/2} \quad \text{---} (20)$$

where $$M_0 = \frac{9}{64}\left(\frac{3\pi}{\alpha^3}\right)^{1/2}\frac{\left(\hbar c/G\right)^{3/2}}{m_p^2}$$

For $M < M_0$ equn (20) dos not have any real solution for R. As mass of dwarf star $M \to M_0$, its radius $R \to 0$. So no dwarf star exists with mass larger than M_0. M_0 is called as Chandrasekhar limit.

Actually Fermi pressure cannot compansate the gravitational pressure for $M > M_0$ and star collapse due to gravitation.

More detailed study gave the chandrasekhar limit as

$$M_0 = \frac{5.75}{\mu_e^2} M_{sun} \quad , \quad M_{sun} = \text{mass of sun} \simeq 2\times10^{33}\, g.$$

and μ_e represents the degree of ionization of gases. In most cases $\mu_e \simeq 2$.

$$\therefore \boxed{M_0 \simeq 1.44\, M_{sun}}$$

References:

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Thank You

For any questions/doubts/suggestions and submission of assignments

Write at E-mail: akgupta@mgcub.ac.in