Lecture-XIV Programme: M. Sc. Physics

Radiation: Radiation from an Arbitrary Source

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Radiation from an Arbitrary Source

❖ In last lecture, we studied the radiation generated by two precise arrangements: oscillating electric dipoles and oscillating magnetic dipoles.

❖ Here we want to use the similar methods to a arrangement of charge and current that is fully arbitrary, apart from that it is localized within some limited volume close to the origin (Fig. 1).

❖ As we know that the retarded scalar potential is:

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}', t - \mathscr{r}/c)}{\mathscr{r}}\, d\tau', \quad \text{------------- [1]}$$

where

$$\mathscr{r} = \sqrt{r^2 + r'^2 - 2\mathbf{r} \cdot \mathbf{r}'}. \qquad \text{------------- [2]}$$

❖ From earlier case, we shall suppose that the field point r is far away, in comparison to the dimensions of the source:

Approximation 1 : $r' \ll r$

(In fact, *r'* is a variable of integration; approximation 1 indicates that the *greatest* value of *r'*, as it varies over the source, is much less than *r*.)

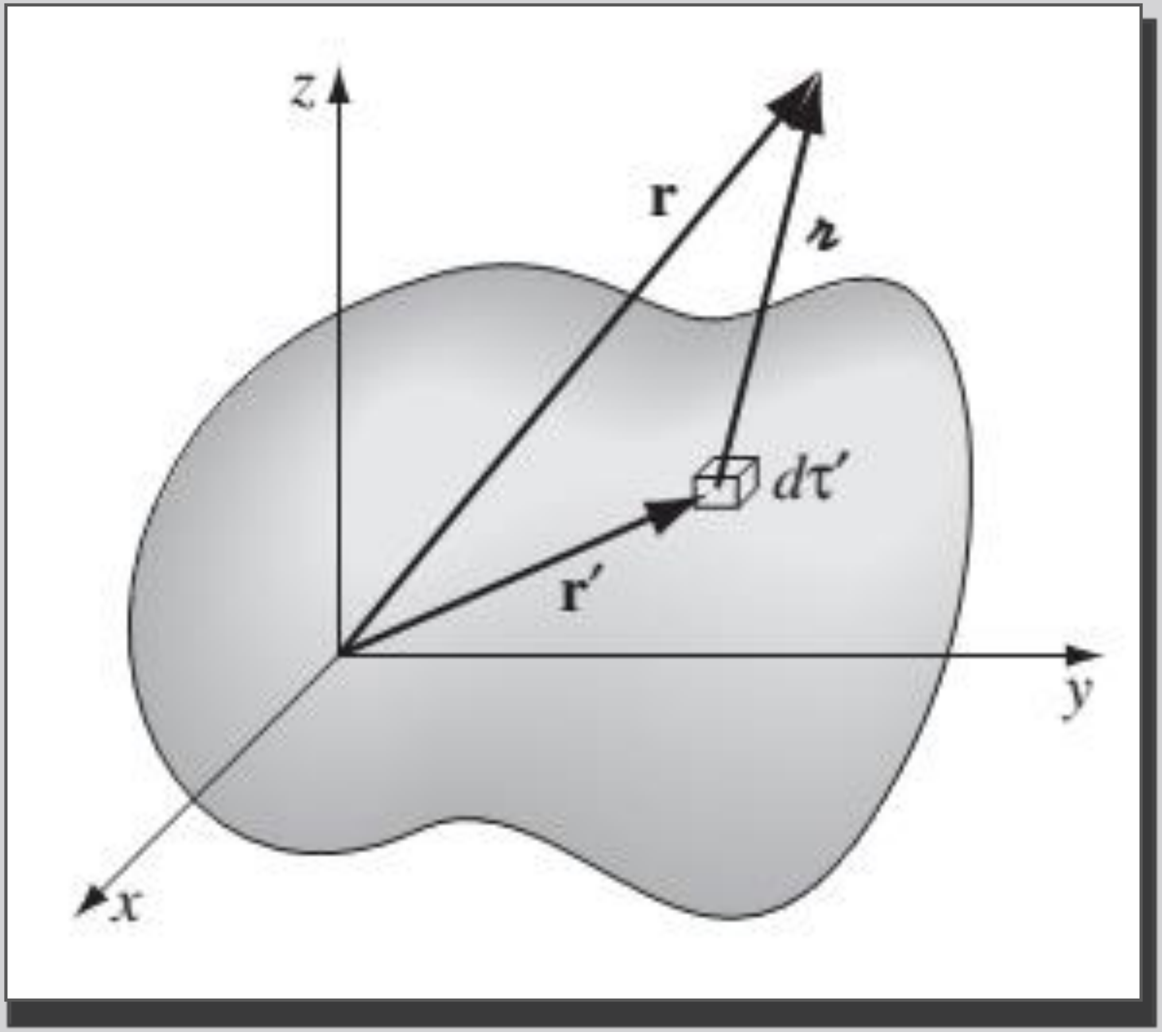

FIGURE 1: Arrangement of an arbitrary charge and current sources within a volume [REF- 1]**

On this statement,

$$\mathscr{r} \cong r\left(1 - \frac{\mathbf{r}\cdot\mathbf{r}'}{r^2}\right),$$

so

$$\frac{1}{\mathscr{r}} \cong \frac{1}{r}\left(1 + \frac{\mathbf{r}\cdot\mathbf{r}'}{r^2}\right)$$ ------------- [3]

and

$$\rho(\mathbf{r}', t - \mathscr{r}/c) \cong \rho\left(\mathbf{r}', t - \frac{r}{c} + \frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}\right).$$ ------------- [4]

❖ **Expanding *ρ* as a Taylor series in *t* regarding the retarded time at the origin,**

$$t_0 \equiv t - \frac{r}{c},$$ ------------- [5]

We get the following form:

$$\rho(\mathbf{r}', t - \imath/c) \cong \rho(\mathbf{r}', t_0) + \dot{\rho}(\mathbf{r}', t_0)\left(\frac{\hat{\mathbf{r}} \cdot \mathbf{r}'}{c}\right) + \dots$$ ------------- [6]

where the dot implies differentiation with respect to time.

❖ The higher order terms in this Taylor expansion would be

$$\frac{1}{2}\ddot{\rho}\left(\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}\right)^2, \quad \frac{1}{3!}\dddot{\rho}\left(\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}\right)^3, \ldots .$$

-------------- [7]

We can afford to leave them, given

Approximation: 2

$$r' \ll \frac{c}{|\ddot{\rho}/\dot{\rho}|}, \quad \frac{c}{|\dddot{\rho}/\dot{\rho}|^{1/2}}, \quad \frac{c}{|\ddddot{\rho}/\dot{\rho}|^{1/3}}, \ldots$$

-------------- [8]

❖ For an oscillating arrangement, each of these ratios is c/ω, and we recover the old approximation 2. In the common situation it is more difficult to understand Eq. 8, however as a procedural issue approximations 1 and 2 amount to keeping just the first order terms in r'.

❖ **Inserting eqs. 3 and 6 into the relation for *V* (eq. 1), and again leaving the second-order term:**

$$V(\mathbf{r},t) \cong \frac{1}{4\pi\epsilon_0 r}\left[\int \rho(\mathbf{r}',t_0)\,d\tau' + \frac{\hat{\mathbf{r}}}{r}\cdot\int \mathbf{r}'\rho(\mathbf{r}',t_0)\,d\tau' + \frac{\hat{\mathbf{r}}}{c}\cdot\frac{d}{dt}\int \mathbf{r}'\rho(\mathbf{r}',t_0)\,d\tau'\right].$$

------------- [9]

❖ **in above equation the first integral is just the total charge, *Q*, at time t_0. As charge is conserved, *Q* is independent of time. The other two integrals correspond to the electric dipole moment at time t_0. Hence-**

$$V(\mathbf{r},t) \cong \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r} + \frac{\hat{\mathbf{r}}\cdot\mathbf{p}(t_0)}{r^2} + \frac{\hat{\mathbf{r}}\cdot\dot{\mathbf{p}}(t_0)}{rc}\right].$$

------------- [10]

❖ **In the static case, the first two terms are denoting to the monopole and dipole contributions to the multi-pole expansion for *V*; the third term, certainly, would not be present.**

❖ **Now, the expression for the vector potential is-**

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t - \mathscr{r}/c)}{\mathscr{r}} \, d\tau'. \qquad \text{-------------- [11]}$$

As you will look in a instant, to first order in *r'* it suffices to replace $\mathscr{r}$ by *r* in the integrand:

$$\mathbf{A}(\mathbf{r}, t) \cong \frac{\mu_0}{4\pi r} \int \mathbf{J}(\mathbf{r}', t_0) \, d\tau'. \qquad \text{-------------- [12]}$$

Note- As the integral of J is the time derivative of the dipole moment, so we can write it as:

$$\int_{\mathcal{V}} \mathbf{J}\, d\tau = d\mathbf{p}/dt,$$

❖ Therefore the expression for the vector potential can be written as:

$$\mathbf{A}(\mathbf{r}, t) \cong \frac{\mu_0}{4\pi} \frac{\dot{\mathbf{p}}(t_0)}{r}.$$

------------- [13]

Similarly,

$$\nabla \times \mathbf{A} \cong \frac{\mu_0}{4\pi r}[\nabla \times \dot{\mathbf{p}}(t_0)] = \frac{\mu_0}{4\pi r}[(\nabla t_0) \times \ddot{\mathbf{p}}(t_0)] = -\frac{\mu_0}{4\pi r c}[\hat{\mathbf{r}} \times \ddot{\mathbf{p}}(t_0)],$$

------------- [14]

❖ Now you observe why it was needless to take the approximation of $\mathscr{r}$ beyond the zeroth order ($\mathscr{r} \cong r$): p is *already* first order in *r′*, and some modifications would be corrections of higher order.

❖ Subsequently we must determine the fields. Once again, we are concerned in the radiation zone (that is, in the fields that stay alive at large distances from the source), thus we continue only those terms that go like 1/*r*:

Approximation 3 : *discard 1/r² terms in E and B*

As, the Coulomb field,

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}}, \quad \text{------------- [15]}$$

arising from the first term in Eq. 9, does not add to the electromagnetic radiation. Indeed, the radiation arises completely from those terms in which we differentiate the argument t_0. It is done as from Eq. 5:

$$\nabla t_0 = -\frac{1}{c}\nabla r = -\frac{1}{c}\hat{\mathbf{r}},$$

------------- [13]

and so

$$\nabla V \cong \nabla\left[\frac{1}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}\cdot\dot{\mathbf{p}}(t_0)}{rc}\right] \cong \frac{1}{4\pi\epsilon_0}\left[\frac{\hat{\mathbf{r}}\cdot\ddot{\mathbf{p}}(t_0)}{rc}\right]\nabla t_0 = -\frac{1}{4\pi\epsilon_0 c^2}\frac{[\hat{\mathbf{r}}\cdot\ddot{\mathbf{p}}(t_0)]}{r}\hat{\mathbf{r}}.$$

------------- [14]

whilst

$$\frac{\partial \mathbf{A}}{\partial t} \cong \frac{\mu_0}{4\pi}\frac{\ddot{\mathbf{p}}(t_0)}{r}. \qquad \text{------------- [15]}$$

Equations for Electric and Magnetic Fields

❖ From Lorentz condition, the expression for the electric field is, thus

$$\mathbf{E}(\mathbf{r}, t) \cong \frac{\mu_0}{4\pi r}[(\hat{\mathbf{r}} \cdot \ddot{\mathbf{p}})\hat{\mathbf{r}} - \ddot{\mathbf{p}}] = \frac{\mu_0}{4\pi r}[\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \ddot{\mathbf{p}})], \qquad \text{------------- [16]}$$

where $\ddot{\mathbf{p}}$ is calculated at time $t_0 = t - r/c$, and

and

$$\mathbf{B}(\mathbf{r}, t) \cong -\frac{\mu_0}{4\pi r c}[\hat{\mathbf{r}} \times \ddot{\mathbf{p}}].$$ ------------- [17]

❖ In particula$\ddot{\mathbf{p}}$, if we utilize spherical polar coordinates, with the *z* axis in the direction of (t_0), then the expressions for the fields can be written as:

$$\mathbf{E}(r, \theta, t) \cong \frac{\mu_0 \ddot{p}(t_0)}{4\pi}\left(\frac{\sin\theta}{r}\right)\hat{\boldsymbol{\theta}},$$ ------------ [18.a]

$$\mathbf{B}(r, \theta, t) \cong \frac{\mu_0 \ddot{p}(t_0)}{4\pi c}\left(\frac{\sin\theta}{r}\right)\hat{\boldsymbol{\phi}}.$$ ------------ [18.b]

Expression for Poynting Vector

- The Poynting vector is defined as:

$$\mathbf{S}(\mathbf{r}, t) \cong \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}) = \frac{\mu_0}{16\pi^2 c}[\ddot{p}(t_0)]^2 \left(\frac{\sin^2\theta}{r^2}\right) \hat{\mathbf{r}},$$ ------------- [19]

Expression for Total Radiated Power

- Now the total power passing through a gigantic spherical surface at radius *r* is

$$P(r, t) = \oint \mathbf{S}(\mathbf{r}, t) \cdot d\mathbf{a} = \frac{\mu_0}{6\pi c}\left[\ddot{p}\left(t - \frac{r}{c}\right)\right]^2,$$ ------------- [20]

Since from previous study we have

$$P_{\text{rad}}(t_0) = \lim_{r\to\infty} P\left(r, t_0 + \frac{r}{c}\right)$$

Finally the total radiated power is

$$P_{\text{rad}}(t_0) \cong \frac{\mu_0}{6\pi c}\left[\ddot{p}(t_0)\right]^2 . \qquad \text{------------- [21]}$$

❖ It is observe that *E* and *B* are mutually perpendicular, normal to the direction of propagation $(\hat{\mathbf{r}})$, and in the ratio *E/ B* = *c*, as always for radiation fields.

Numerical problems

1. A parallel-plate capacitor C, with plate separation d, is given an initial charge (±) Q 0. It is then connected to a resistor R, and discharges, Q(t) = Q 0e−t/ RC.

   (a) What fraction of its initial energy $(Q_0^2/2C)$ does it radiate away?
   (b) If *C* = 1 pF, *R* = 1000 , and *d* = 0.1 mm, what is the actual *number*? In electronics we don't ordinarily worry about radiative losses; does that seem reasonable, in this case?

2. A current I(t) flows around the circular ring. Derive the general formula for the power radiated, expressing your answer in terms of the magnetic dipole moment, m(t), of the loop.

References:

1. *Introduction to Electrodynamics, David J. Griffiths*

2. *Modern Electrodynamics, Andrew Zangwill*

3. *Elements of Electromagnetics, 2nd edition by M N O Sadiku*

4. *Engineering Electromagnetics, W H Hayt and J A Buck.*

5. *Elements of Electromagnetic Theory & Electrodynamics, Satya Prakash*

- **For any query/ problem contact me on whatsapp group or mail on me**

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- **Next *** we will discuss the Radiation: Power Radiated by a Point Charge and numerical problems.**

Thank you